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From symmetries of partial differential equations towards secondary ("quantized") calculus

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Abstract

Diffieties are analogues of algebraic varieties for partial differential equations. They are a kind of (generally, infinite-dimensional) manifolds supplied with an infinite-order contact structure. Secondary, or more speculatively, "quantized" calculus arises as a sort of differential calculus over filtered smooth function algebras on diffieties that respects the contact structure. This paper, written as an informal introduction and invitation to Secondary Calculus, is an account of the author's attempt to understand what should be the analogue of the Schrödinger equation for quantum field theory. So, more attention is paid to motivations than to exact constructions and formulas.

Keywords: partial differential equations, jet spaces, symmetries of partial differential equations, diffieties, $\mathscr{C}$-spectral sequence, secondary differential operators and differential forms, solution singularities, "troubled times"

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0. Introduction

> But science is not yet just a catalogue of ascertained facts about the universe; it is a mode of progress, sometimes tortuous, sometimes uncertain. And our interest in science is not merely a desire to hear the latest facts added to the collection, we like to discuss our hopes and fears, probabilities and expectations.
>
> Sir A. Eddington

The pre-history of rational mechanics was the study of the so-called simple mechanisms. A number of attempts to explain the whole Nature as a machine composed of these mechanisms was made in that period. The "standard schemes" and "models" of the modern quantum field theory (QFT) look very much like these simple mechanisms.

This analogy, maybe, makes clear the reasons of the almost common feeling that quantum field theory in its present form is not yet a "true" well-established theory. Below we undertake an attempt to analyse why this is so and what ingredients are to be added to the solution to get the desired crystallization.

Having this in mind we start the paper with some general observations on the genesis of long-scale theories. These introductive pages furnish our subsequent considerations with the necessary initial impulse. Following it we eventually arrive at Secondary or, more speculatively, Quantized Differential Calculus, which seems to have some chances to provide the passage from "standard models" to the "true" theory with the necessary mathematical background.

We would like to stress from the very beginning that Secondary Calculus is only a language on which, we hope, QFT can be developed smoothly, i.e. without "renormalizations", "anomalies", etc. If it is so, the fundamental problem to translate QFT systematically into Secondary Calculus remains to be carried out separately. Of course, results and experiences accumulated up to now in the study of concrete models are indispensable to this purpose.

This paper is neither a review nor a research account but a long motivation for this Secondary Calculus. We describe informally some principal ideas and results already obtained in this field and also indicate some problems and perspectives which seem promising at this moment.

It was not our intention to present here a systematic and rigorous exposition of Secondary Calculus. That would be hardly possible within the limits of one, even long, paper. So, we restrict ourselves to a general panorama which could help the interested reader to enter the subject by consulting the attached bibliography. Details and techniques completing this text can be found in Refs. [51,53,49], which we suggest to read first. They should be followed by Refs. [1,17,50, 38,54,40,41,15,16,52,28].

And, finally, the first "philosophical" pages of this paper are to be read semi-seriously keeping an eye on the uncertainty principle: a superfluous making more precise the meaning of words used there will kill the motivating impulses the author hopes they emit.

1. From symmetries to conceptions

It is banal to say that every theory has its origin in rather simple things. But what are they? The word "simple" of common language incorporates many meanings. In linear approximation they can be displayed by the following diagram:

$$\boxed{\text{BANAL}} - \boxed{\;\cdot\;} - \cdots - \boxed{\;\cdot\;} = \boxed{\underline{\text{SYMMETRIC}}}$$

in which the dots indicate the "intermediate states". In other words, we find enough reasons to interpret "symmetric" as "simple but not banal". Details are just obstructions to symmetry. So, the models manifesting only the essence of the phenomena in question are necessarily symmetric. Recall euclidean geometry, Copernicus' planetary system, Newton's laws in mechanics or special relativity to illustrate this idea. Hence, we accept as the leading principle that the initial stage in the genesis of theories is the study of symmetric models. (Of course, the above remarks are applicable only to rather long-scale situations.)

Symmetry considerations replace quite well the conceptual thinking in studying symmetric models. This is why they work well at first, especially for mathematically based theories, owing to the fact that "symmetry" implies "solvability" and "integrability" in this case.

At this point the theory passes to the next stage in its development when the dominating paradigm states that everything can be composed of simple (symmetric) elements studied earlier and the only thing to be understood is how. Schematically, this period can be characterized as the time when operative conceptions of the future "true" theory, not yet discovered, are substituted for their "morphemes" and when more or less mechanical mosaics of the latter replace the calculus of these conceptions. This is the reason to call this stage "morphological".

A serious deficiency of these morphological compositions is that many of them are to be corrected constantly to be in agreement with new experimental data and theoretical demands. This produces numerous perturbation-like schemes which are very characteristic of the morphological era.

Ptolemy's planetary system with its numerous epi- and hypo-cycles and quantum electrodynamics with its renormalizations illustrate this quite well. Also, one can learn from these examples that the incredibly exact correspondence to experiments is not all that is needed to be a "true" theory. Of course, it is nothing bad to use a perturbation scheme for technical purposes. But it would be hardly reasonable to erect a skyscraper on a perturbative foundation.

Afterwards theories enter their "troubled times", or, which is better to say, the stage of conceptual self-organization. Surely, this is the longest, mysterious and even dramatic period in the birth process of a new theory. At that time some hidden selection mechanisms acting in the relevant scientific community draw out, step by step, the necessary new conceptions and one day it appears that they constitute that unique language in which the laws of the scope of the phenomena in question can be expressed quite adequately and, therefore, elegantly. This the just the birthday of a new theory.

Darwin's selection theory seems to be applicable to this selection of conceptions as well. For example, one can see many fantastic creations appearing during troubled times (for example, look at the history of QFT 23 years back). This is typical for situations when the expressive powers of the language do not correspond to the subject to be described.

Summing up we represent our idea concerning the genesis of mathematically based theories by the scheme

$$\boxed{\begin{array}{c}\text{symmetric origin}\\ \text{(``beautiful times'')}\end{array}} = \boxed{\begin{array}{c}\text{morphological}\\ \text{era}\end{array}}$$
$$= \boxed{\begin{array}{c}\text{conceptual self-organization}\\ \text{(``troubled times'')}\end{array}} = \boxed{\begin{array}{c}\text{conceptual}\\ \text{happy end}\end{array}} \qquad (1)$$

Of course, in reality, the indicated periods get mixed and this can happen, sometimes, in a very curious way. For example, nowadays synthetic geometries, typical creations of the morphological era, have almost left the land, being substituted by differential geometry. On the other hand, measure theory, being a morphological realization of the idea of integration, coexists peacefully with its future conqueror, namely, the de Rham-like cohomologies.

The passage from attempts to model the scope of new phenomena in terms of the "old", already existing mathematical language to a new one of a higher level, whose expressive potentials are just adequate to the new demands, is the essence of scheme (1). Here we use "mathematical language" in the spirit of "program language". This enables us to take into account anthropomorphic elements present implicitly in the theories due to the fact that individual brains and scientific communities are something like computers and computer networks, respectively. The history of metric geometry from its Hellenistic symmetric form based on common logic up to its modern Riemannian form based on Calculus gives an ideal illustration of the above scheme.

2. "Troubled times" of quantum field theory

Assuming scheme (1) to be true its becomes quite clear that nowadays QFT passes through its "troubled times". Even some key words of QFT's current vocabulary, such as "renormalizations", "broken symmetries", "anomalies", "ghosts" etc., indicate a deep discrepancy of its physical content and the mathematical equipment used. Also, one can see too many Lie groups, algebras, etc. up to quantum and quasi-quantum ones, and symmetry considerations based on them, which play a fundamental role in the structure of modern QFT. This shows that the theory is not far enough from its symmetric origin. In fact, the strongest and most obvious argument in favor of these "troubled times" comes from the perturbation type structure of the existing theory. However, the absence of real alternatives and long-time habits have reduced the value of this argument almost to zero.

The author realizes that the sceptic reader, even convinced of these "troubled times", will prefer to follow the current research activity in expectation of times when the aforementioned natural selection mechanisms will have accomplished their work. So, this paper is mainly dedicated to those who would be interested to seek some possible artificial selection mechanisms, which, as is well known, work much faster.

At this point we pass to look for this "program language" for QFT, being motivated by the above "evolution theory". Of course, the latter should be exposed with more details to be perceived correctly. But we do not take the risk to go more in this direction, remembering the attitude toward any philosophy at the end of the "point theoretic" epoch we are living in. Instead we invite the reader to return to this point once again after having read the whole paper. Also, a development of the above general ideas can be found in Ref. [1], ch. 1. In particular, there we touch such topics as which anthropomorphic factor stands behind the idea to put set theory in the foundations of the whole mathematics and why, properly, Calculus is the language of classical physics.

3. "Linguization" of the Bohr correspondence principle

We find the initial data in the following two general postulates, which seem to be beyond doubt:

I. Calculus is the language of classical physics.

II. Classical mechanics is the limit case with $h \to 0$ of quantum mechanics ("the Bohr correspondence principle").

These are our initial position and momentum, respectively.

To avoid misunderstanding we would like to stress that the word "Calculus" is used here, and later on, in its direct sense, i.e. as a system of conceptions (say, vector fields, differential forms, differential operators, jets, de Rham's, Spencer's, ... cohomologies, etc., governed by general rules, or formulas like $d^2=0$, $L_X = i_X \circ d + d \circ i_X$, etc.). As we have shown in Ref. [42], they all constitute a sort of "logic algebra" due to the fact that differential calculus can be, in fact, developed in a purely algebraic way over an arbitrary (super-) commutative algebra A (see also Ref. [17], ch. 1). This algebraically constructed Calculus coincides with the standard one for smooth function algebras $A = C^\infty(M)$. Also, one can learn from this algebraic approach, and this is very important to emphasize, that there are many things to discover and to perceive in order to close this logic algebra, i.e. to get the whole Calculus. Higher-order analogs of the de Rham complexes [48] give such a kind of example.

Thus, the first postulate suggests to look for an extension of Calculus while the second one defines more precisely the direction to aim at. Having this in mind we need to extract the mathematical essence of Bohr's correspondence principle and the following diagram illustrates how it can be done:

$$\begin{array}{ccc}
\boxed{\begin{array}{c}\text{equations of quantum}\\ \text{mechanics}\\ \text{(Schrödinger's equations)}\end{array}} & \underset{\text{quantization}}{\overset{\overline{h \to 0}}{\rightleftarrows}} & \boxed{\begin{array}{c}\text{equations of classical}\\ \text{mechanics}\\ \text{(Hamilton's equations)}\end{array}} \\
 & & \Downarrow \text{"mathematization"} \\
\boxed{\begin{array}{c}\text{partial differential}\\ \text{equations}\end{array}} & \xrightarrow{\underline{\text{charQ}}} & \boxed{\begin{array}{c}\text{ordinary differential}\\ \text{equations}\end{array}}
\end{array}$$

Here CHAR_Σ denotes the map which assigns to a given system of partial differential equations (p.d.e.) $\mathscr{Y}$ a system $\mathscr{Y}^0_\Sigma$ or ordinary equations describing how Σ-type singularities of solutions of $\mathscr{Y}$ propagate. What is meant by solution singularity types Σ and what is, in particular, the above singularity type Q will be discussed later on, see also Refs. [43,52,26,28]. But now we will explain what are the reasons for suspecting CHAR_Q to be behind the Bohr correspondence principle.

First, note that the mathematical background of the passage from wave to geometrical optics can be naturally presented in the form $\mathscr{Y} \to \mathscr{Y}^0_{\mathrm{FOLD}}$, where FOLD stands for the folding type singularity of multivalued solutions of $\mathscr{Y}$ (see section 16). On the other hand, multivaluedness of solutions is related to non-uniqueness of the Cauchy problem and, therefore, to the theory of (bi-)characteristics.

Remark. There exists a dual way to pass to geometrical optics proposed by Luneburg [25] and based on the study of discontinuous solutions. However, the choice of Luneburg's approach instead of that we have adopted does not lead us to essential changes in our subsequent arguments.

Second, remembering that Schrödinger discovered his famous equations proceeding from the analogy with wave-geometric optics, one can expect a similar mechanism in the passage from quantum to classical mechanics [36]. More precisely, it seems natural to hypothesize the equations of classical mechanics to be the Q-characteristic equations of the corresponding equations of quantum mechanics. These hypothetical Q-characteristic equations should play a similar role with respect to an appropriate "quantum" solution singularity type as the standard characteristic equations do with respect to the singular Cauchy problem. This hypothesis becomes almost evident in the framework of Maslov's approach to quasi-classical asymptotics [30]. We refer also to the lectures by Levi-Cività [20] and the work by Racah [35], one of the first attempts to go this way.

This all motivates us to take the formula

$$\mathrm{QUANTIZATION} = \mathrm{CHAR}_Q^{-1} \tag{2}$$

as the leading principle and we go to seek its consequences.

First of all, the direct attempt to extend (2) to QFT leads immediately to the problem illustrated by the following diagram:

$$\begin{tikzcd}
\boxed{\text{quantum fields}} \arrow[r, "\hbar\to 0"] \arrow[d, Rightarrow] & \boxed{\text{classical fields}} \arrow[d, Rightarrow] \\
\boxed{?} \arrow[r, "\mathrm{CHAR}_Q"] \arrow[ur, phantom, "\text{“mathematization”}"] & \boxed{\begin{array}{c}\text{partial differential}\\ \text{equation}\end{array}}
\end{tikzcd} \tag{3}$$

In other words, we have to answer the question: what kind of mathematical objects are to be placed into the left lower rectangle of (3) or, more precisely, what is the mathematical nature of the equations whose solution singularity propagation is described by means of partial differential equations? The scheme

$$\boxed{\begin{array}{c}\text{ordinary differential}\\ \text{equations}\end{array}} \xleftarrow{\text{CHAR}} \boxed{\begin{array}{c}\text{partial differential}\\ \text{equations}\end{array}} \xleftarrow{\text{CHAR}} \boxed{\;?\;}$$

motivates us to call these, yet unknown, mathematical objects *secondary quantized differential equations.*

Thus, the problem to consider next is

What are secondary quantized differential equations? (4)

All the preceding discourses do not furnish us with the necessary impulse to attack it. In searching such an impulse we consider the simplest situation when a CHAR-type mapping appears:

$$\sum_i a_i(x) \frac{\partial u}{\partial x_i} = b_i(x) \xrightarrow{\text{CHAR}} \{x_i = a_i(x)\} .$$

In other words, we will examine the passage from vector fields to ordinary differential equations making an attempt at understanding what secondary ("quantized") vector fields should be.

We can profit from the simplicity of this situation that comes from the symmetry of the context in full accordance with section 1. More exactly, infinitesimal symmetries of the system $x_i = a_i(x)$ are vector fields $Y = \sum c_i(x)\ \partial/\partial x_i$ commuting with the field $X = \sum a_i(x)\ \partial/\partial x_i$ and, as is well known, any vector field admits locally a plot of fields commuting with it. For our purposes it is important to observe that symmetries of the system $x_i = a_i(x)$ are objects of the same nature as the differential operator (namely, $X = \sum a_i(x)\ \partial/\partial x_i$) defining the first-order part of the initial equation $X(u) = b$. For this reason it seems very likely that secondary quantized vector fields are identical to symmetries of partial differential equations. So, proceeding to check this hypothesis we have to answer the question in the following title.

4. What are symmetries of partial differential equations and what are partial differential equations themselves?

It is not to so easy to answer conceptually both these questions. To see why this is so let us consider a partial differential equation, say,

$$F(x, u, u_i, \ldots, u_{(k)}) = 0 , \tag{5}$$

where $x=(x_1, ..., x_n)$, $u=(u^1, ..., u^m)$ and $u_{(l)}$ stands for the totality of all lth order derivatives of the dependent variables u with respect to the independent ones x, and ask what the symmetries of (5) are. If one treats (5) to be a relation between two different groups of variables, dependent and independent ones, then it comes naturally to define a symmetry of (5) to be a transformation of the dependent variables and, separately, of the independent ones which preserves this relation. More exactly, it is proposed to call symmetries of (5) transformations of the form

$$(x, u) \mapsto (\bar{x}(x), \bar{u}(u)) \tag{6}$$

which, being extended canonically to derivatives, preserve (5).

This historically first approach was followed by the following two ones enlarging the group of transformations (6), first, to

$$(x, u) \mapsto (\bar{x}(x), \bar{u}(x, u))$$

and then to

$$(x, u) \mapsto (\bar{x}(x, u), \bar{u}(x, u)) .$$

Unfortunately, all these definitions are based on an "ad hoc" choice of transformations to be taken as symmetries. So, going this way one can never be sure that a new "Ansatz", like (6) or (7), is the "true" final definition. For example, the transformations (7) were, historically, followed by the famous contact transformations of Lie. Namely, he proposed to consider as symmetries of (5) transformations of the form

$$(x, u, u_x) \mapsto (\bar{x}, \bar{u}, \bar{u}_{\bar{x}}) ,$$

where $\bar{x}=\bar{x}(x, u, u_x)$, $\bar{u}=\bar{u}(x, u, u_x)$, $\bar{u}_x=\bar{u}_x(x, u, u_x)$, which preserve (5) together with the equation $\mathrm{d}u-\sum u_{x_i}\,\mathrm{d}x_i=0$.

But why cannot somebody find something else? Lie himself had not answered this question. But he understood deeply that an expression of the form (5) is not a sovereign object to be transformed directly but only a label of it. In particular, his discovery of contact transformations was based on a geometric interpretation of what stands behind labels of the form $f(x, u, u_x)=0$ (for a discussion see Ref. [51]).

In general, it is clear that symmetries of a mathematical object are to be its invertible morphisms ("transformations") into itself. So, one can see from the above discussion that the observed difficulties to define symmetries of p.d.e.'s come from the fact that really we do not know what, properly, partial differential equations are.

The last question is, in fact, neither to absurd nor so innocent as it may appear at first glance. On the contrary, having answered it we will gain much more than the true concept of symmetry for p.d.e.'s.

In what follows we allow ourselves, sometimes, to call labels what is commonly called partial differential equations, i.e. expressions of the form (5). This is to underline the difference between the common and the conceptual use of these last words.

Now we pass to the necessary preliminaries.

5. Jets

Note that labels of differential equations are "algebraic" relations between independent variables and derivatives of the latter up to a prescribed order. So, each label defines a submanifold, one of the possible local charts of which is formed by all these variables and derivatives. These manifolds are called jet spaces (or manifolds) and we are interested in them to reinterpret labels in an invariant coordinate-free form. Below the necessary definitions and elementary facts on jets are collected.

Let us fix an $(n+m)$-dimensional manifold E, $m\geqslant 0$, and a non-negative integer n. This is to emphasize our intention to consider n-dimensional submanifolds of E. If L, L' are two of them and $a\in L\cap L'$ we say that they have the same kth order jet at $a\in E$ if they are tangent to each other up to order k at a. So, k-jets are equivalence classes of n-dimensional submanifolds of E with respect to the relation "be k-tangent". We denote by $[L]_a^k$ the kth order jet of the n-dimensional submanifold $L\subset E$ at $a\in L$. The totality of kth order jets of all possible n-dimensional submanifolds $L\subset E$ at all points $a\in E$ is denoted by $J^k(E, n)$. Being supplied with a natural structure of smooth manifold, this is called the kth order jet space of n-dimensional submanifolds of E.

Remark. If E has a fibred structure, say, $\pi: E\to M$, $\dim M=n$, then one can consider a special class of n-dimensional submanifolds of E that are graphs of local sections of π. The kth order jets of these graphs constitute an open and everywhere dense subset of $J^k(E, n)$, called the kth order jet space of sections of π and denoted by $J^k\pi\subset J^k(E, n)$.

Example 1 ($k=0$). Evidently, every two n-dimensional submanifolds of E passing through a point $a\in E$ are tangent to each other with order zero at a. Therefore, there exists only one 0-jet at a and one can identify $J^0(E, n)$ with E.

Example 2 ($k=1$). Two n-dimensional submanifolds of E are tangent to each other with order one at the point a iff they have the same tangent space at a. So, first-order jets at $a\in E$ can be identified with n-dimensional linear subspaces of the tangent space T_aE of E at a.

We stress here that all above definitions are also valid for $k=\infty$. In particular, $J^\infty(E, n)$ is well defined as a set. However, it requires some care to impose a smooth structure on it. This will be done below.

Natural maps

$$\alpha_{k,l}: J^k(E, n) \to J^l(E, n) , \quad [L]_a^k \mapsto [L]_a^l$$

with $\infty \geqslant k \geqslant l \geqslant 0$ unite together jet spaces of different order into a family. In particular, they form the sequence

$$E=J^0(E, n) \xleftarrow{\alpha_{1,0}} J^1(E, n) \xleftarrow{\alpha_{2,1}} \cdots \xleftarrow{\alpha_{k,k-1}} J^k(E, n) \xleftarrow{\alpha_{k+1,k}} \cdots , \qquad (8)$$

the inverse limit of which coincides with $J^\infty(E, n)$.

For a given $L \subset E$, $\dim L=n$, we have the map

$$j_k(L): L \to J^k(E, n) , \quad L \ni a \mapsto [L]_a^k \in J^k(E, n) .$$

A function φ on $J^k(E, n)$ is said to be *smooth* iff $\varphi \circ j_k(L) \in \mathrm{C}^\infty(L)$ for every n-dimensional $L \subset E$. We define in this way the smooth function algebra on $J^k(E, n)$ and, therefore, a smooth manifold structure on it. We remark that this definition works as well for $k=\infty$.

It follows directly from the definitions that $j_k(L)$ is a smooth map, and

$$L_{(k)} := \mathrm{im}\, j_k(L) \subset J^k(E, n)$$

is an n-dimensional smooth submanifold of $J^k(E, n)$. Also the identities

$$j_l(L) = \alpha_{k,l} \circ j_k(L) , \quad k \geqslant l ,$$

show that $\alpha_{k,l}$ is a smooth surjection and

$$\alpha_{k,l}^*: \mathrm{C}^\infty(J^l(E, n)) \to \mathrm{C}^\infty(J^k(E, n))$$

is an imbedding. The direct limit of the following sequence of monomorphisms:

$$\mathrm{C}^\infty(E) \xrightarrow{\alpha_{1,0}^*} \mathrm{C}^\infty(J^1(E, n)) \xrightarrow{\alpha_{2,1}^*} \cdots \xrightarrow{\alpha_{k,k-1}^*} \mathrm{C}^\infty(J^k(E, n)) \xrightarrow{\alpha_{k+1,k}^*} \cdots$$

coincides, evidently, with the smooth function algebra of $J^\infty(E, n)$. We adopt the standard notation $\mathrm{C}^\infty(J^\infty(E, n))$ for it.

It is convenient to shorten these long notations as follows:

$$\mathscr{F}_k = \mathscr{F}_k(E, n) := \mathrm{C}^\infty(J^k(E, n)) , \quad 0 \leqslant k < \infty ,$$

$$\mathscr{F} = \mathscr{F}(E, n) := \mathrm{C}^\infty(J^\infty(E, n)) .$$

We get the filtration

$$\mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k \subset \cdots \subset \mathscr{F}$$

of the algebra $\mathscr{F}$ by identifying $\mathscr{F}_k$ with its image in $\mathscr{F}$ under the monomorphism $\alpha_{k,\infty}^*$. The differential calculus over the filtered algebra $\mathscr{F} = \{\mathscr{F}_k\}$ gives the necessary rigorous foundations for all our subsequent constructions (see Ref. [17]).

In particular, it enables us to handle $J^\infty(E, n)$ in many aspects as a usual finite-dimensional smooth manifold.

A local chart on E is called *divided* if the coordinate functions forming it are divided into two parts consisting of n and m functions, respectively. The first of them, say $x_1, ..., x_n$, are called "*independent*" and the second ones, say $u^1, ..., u^m$, "*dependent*" variables.

A divided chart on E generates a local chart on $J^k(E, n)$, $0 \leqslant k \leqslant \infty$, which consists of the functions

$$x_1, ..., x_n, u^1, ..., u^m, ..., u^i_\sigma, ..., \quad |\sigma| \leqslant k ,$$

where σ stands for a multi-index, say $\sigma = (i_1, i_2, ..., i_n)$, and $|\sigma| = i_1 + \cdots + i_n$. The function u^i_σ is defined by the condition

$$u^i_\sigma \circ j_k(L) = \frac{\partial^{|\sigma|} f^i}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} , \quad \forall L \subset E, \dim L = n,$$

where

$$u^i = f^i(x_1, ..., x_n) , \quad i = 1, ..., m , \tag{9}$$

are local equations of L. And now we see that the jet manifolds $J^k(E, n)$ (or $J^k\pi$) are exactly those ones which naturally carry coordinate systems composed of independent variables together with their derivatives.

A standard label of a system of partial differential equations looks as

$$F_j(x, u, ..., u^i, ...) = 0 , \quad j = 1, ..., l , \tag{10}$$

where $x = (x_1, ..., x_n)$, $u = (u^1, ..., u^m)$. It is natural to interpret (10) as local equations of a submanifold $\mathscr{Y} \subset J^k(E, n)$ for a suitable E. For instance, $u_t = u_{xx} + uu_x$ defines the hypersurface

$$u_{(2,0)} + uu_{(1,0)} - u_{(0,1)} = 0$$

in $J^3(\mathbb{R}^3, 2)$ where $\mathbb{R}^3 = \{(x, t, u)\}$ and $x_1 = x$, $x_2 = t$, $u^1 = u$ is the divided coordinate system in it.

This motivates the following coordinate-free version of the standard definition of partial differential equations.

Definition. A submanifold $\mathscr{Y} \subset J^k(E, n)$ is called *a label of a system of partial differential equations* imposed on n-dimensional submanifolds of the given manifold E.

Let a submanifold $L \subset E$ be given by (9). Then the functions $f^i(x)$, $i = 1, ..., m$, satisfy the system (10) iff $L_{(k)} \subset \mathscr{Y}$. So, the manifold $L_{(k)}$ can be taken as the coordinate-free version of the notion of solution.

We remark that $J^k\pi$ can replace $J^k(E, n)$ in all the previous discussions.

6. Higher-order contact structures

Jet spaces possess by birth a natural geometrical structure, namely, the so-called *kth order contact structure*, or the *Cartan* distribution. This means that a linear subspace, say C_θ, of the tangent space $T_\theta J^k(E, n)$ is assigned to each point $\theta \in J^k(E, n)$.

The subspace C_θ can be defined as follows. First, introduce some special n-dimensional subspaces of $T_\theta J^k(E, n)$, called *R-planes*. By definition an R-plane at $\theta \in J^k(E, n)$ is a subspace of the form $T_\theta L_{(k)}$ supposing that $\theta = [L]_a^k$. We stress here that not every n-dimensional subspace of $T_\theta J^k(E, n)$ is an R-plane and more than one R-plane pass through θ if $k < \infty$ and $mn > 0$. Second, put

$$C_\theta = \{\text{the linear envelope of all } R\text{-planes at } \theta\} .$$

A simple computation shows that

$$\dim J^k(E, n) = \dim T_\theta J^k(E, n) = m\binom{n+k}{k} + n , \quad 0 \leqslant k < \infty$$

and

$$\dim C_\theta = m\binom{n+k-1}{k} + n , \quad 0 \leqslant k < \infty .$$

In particular, $\dim C_\theta \to \infty$ with $k \to \infty$.

Example. (classical contact structure). Consider the manifold $J^1(\mathbb{R}^{n+1}, n)$. In this case $m = k = 1$, $\dim J^1(\mathbb{R}^{n+1}, n) = 2n+1$, $\dim C_\theta = 2n$, i.e., C_θ is a hyper-plane in $T_\theta J^1(\mathbb{R}^{n+1}, n)$. Dividing standard cartesian coordinates $(x_1, ..., x_{n+1})$ in $\mathbb{R}^{n+1}$ as $x = (x_1, ..., x_n)$, $u = x_{n+1}$ we get the local coordinates $(x_1, ..., x_n, u, p_1, ..., p_n)$ in $J^1(\mathbb{R}^{n+1}, n)$, where $p_i = u_{x_i}$. In these coordinates C_θ is given by the equation

$$\mathrm{d}u - \sum_{i=1}^{n} p_i \, \mathrm{d}x_i = 0 ,$$

in which one can recognize the classical (= first-order) contact structure of Lie.

In the general case C_θ is given by the system

$$\mathrm{d}u^i - \sum_{j=1}^{n} u^i_{\sigma+1_j} \, \mathrm{d}x_j = 0 , \quad 1 \leqslant i \leqslant m , \quad 0 \leqslant |\sigma| < k , \tag{11}$$

where $\sigma + 1_j$ stands for multi-index $(i_1, ..., i_j + 1, ..., i_n)$ supposing that $\sigma = (i_1, ..., i_j, ..., i_n)$.

The common feature of all kth-order contact structures, $0 < k < \infty$, is that they all are, in a sense, "completely non-integrable" distributions. In contrast, the infinite-order contact structure on $J^\infty(E, n)$ is completely integrable. This is easily seen from the fact that the Pfaff system (11) satisfies the Frobenius complete integrability conditions if $k=\infty$. Moreover, the Cartan distribution on infinite-order jet spaces is finite dimensional in spite of its being the inverse limit of the Cartan distributions on finite-order jet spaces whose dimension grows infinitely with the order of jets. In fact, the dimension of this distribution is equal to n, i.e., $\dim C_\theta=n$ for $\theta\in J^\infty(E, n)$. This means that there exists only one R-plane for any $\theta\in J^\infty(E, n)$, which coincides automatically with C_θ.

All these remarks are to stress that infinite-order jet spaces differ from finite-order ones not only in that they are infinite dimensional. It looks paradoxical but the former are simpler objects than the latter with regard to the properties of basic structures they carry naturally.

7. Differential equations are diffieties

Now using infinite jet spaces we can answer the question: what are differential equations?

Let

$$D_i=\frac{\partial}{\partial x_i}+\sum_{k,\sigma} u^k_{\sigma+1_i}\frac{\partial}{\partial u^k_\sigma} \tag{12}$$

be the so-called *total derivative operator* with respect to x_i. Then the system

$$F_j=0\,,\quad D_iF_j=0\,,\qquad 1\leqslant j\leqslant l\,,\quad 1\leqslant i\leqslant n\,, \tag{13}$$

defines the first extension of (10). Evidently, the system (13) is equivalent to (10) in the sense that both have the same set of solutions. The system (13), in fact, is a local coordinate description of a new label $\mathcal{Y}_{(1)}\subset J^{k+1}(E, n)$, which can be defined, purely geometrically, in terms of the contact structure on $J^k(E, n)$ only. More generally, one can define *the sth extension* $\mathcal{Y}_{(s)}\subset J^{k+s}(E, n)$ of the label $\mathcal{Y}\subset J^k(E, n)$ by means of the system

$$F_j=0,\quad ...,\quad D_\sigma F_j=0, ...,\qquad |\sigma|\leqslant s\,, \tag{14}$$

with $D_\sigma=D_1^{i_1}\circ\cdots\circ D_n^{i_n}$ where $\sigma=(i_1, ..., i_n)$. In this definition s can be taken equal to infinity. In this case we write $\mathcal{Y}_\infty$ instead of $\mathcal{Y}_{(\infty)}$. Clearly, $\mathcal{Y}_\infty\subset J^\infty(E, n)$. It is not difficult to see that

$$\alpha_{k+s,k+t}(\mathcal{Y}_{(s)})\subset\mathcal{Y}_{(t)}$$

for every $s\geqslant t$. In particular, we have the following sequence

$$\mathcal{Y}=\mathcal{Y}_{(0)}\xleftarrow{\alpha_{k+1,k}}\mathcal{Y}_{(1)}\xleftarrow{\alpha_{k+2,k+1}}\cdots\leftarrow\mathcal{Y}_{(s)}\xleftarrow{\alpha_{k+s+1,k+s}}\cdots$$

the inverse limit of which coincides with $\mathcal{Y}_\infty$.

We define the smooth function algebra $\mathcal{F}(\mathcal{Y}_\infty)$ on $\mathcal{Y}_\infty$ to be the restriction of the smooth function algebra of $J^\infty(E, n)$ on $\mathcal{Y}_\infty$:

$$\mathscr{F}(\mathscr{Y}_\infty) := \mathscr{F}(E, n)\,|_{\mathscr{Y}_\infty} .$$

This algebra is filtered naturally by its subalgebras $\mathscr{F}_s(\mathscr{Y}_\infty)$, $s=0, 1, 2, ...$, where $\mathscr{F}_s(\mathscr{Y}_\infty)$ is the pull-back of the algebra $C^\infty(\mathscr{Y}_{(s)})$ via the map $\alpha_{\infty,k+s}: \mathscr{Y}_\infty \to \mathscr{Y}_{(s)}$,

$$\mathscr{F}_0(\mathscr{Y}_\infty) \subset \mathscr{F}_1(\mathscr{Y}_\infty) \subset \cdots \subset \mathscr{F}_s(\mathscr{Y}_\infty) \subset \cdots \subset \mathscr{F}(\mathscr{Y}_\infty) .$$

This enables us to develop the necessary pithy differential calculus over $\mathscr{Y}_\infty$ understood as the calculus over the filtered commutative algebra $\mathscr{F}(\mathscr{Y}_\infty) = \{\mathscr{Y}_s(\mathscr{Y}_\infty)\}$ (see Ref. [17]).

Also, $\mathscr{Y}_\infty$ inherits the infinite-order contact structure from the ambient space $J^\infty(E, n)$. More exactly, if $\mathscr{Y}$ is a formally integrable system and $\theta \in \mathscr{Y}_\infty$, then $C_\theta \subset T_\theta(\mathscr{Y}_\infty)$. Thus, $\mathscr{Y}_\infty$ is supplied canonically with a "contact structure" which is an n-dimensional distribution on it satisfying the Frobenius complete integrability conditions. Manifolds of the form $\mathscr{Y}_\infty$ considered together with the contact structure described above are local forms of the objects which we call diffieties:

Definition. A manifold $\mathscr{O}$ supplied with an n-dimensional distribution is called a *diffiety* if it is locally of the form $\mathscr{Y}_\infty$.

The number n is called the *diffiety dimension* of $\mathscr{O}$ and is denoted by Dim $\mathscr{O}$. Of course, it differs, generally, from the usual dimension of $\mathscr{O}$, which is equal to infinity as a rule.

Now we take as the leading principle that, conceptually,

systems of partial differential equations are diffieties .

In particular, $\mathscr{Y}_\infty$ is the object which stands behind the label $\mathscr{Y} \subset J^k(E, n)$ [or behind (10)].

It would be too naive to try to change the long-time established terminology by using the words "differential equations" in their new meaning, i.e. as a synonym of "diffiety". For this reason we retain their traditional meaning, remembering, however, that this is simply referring to a label. So, these words are to be substituted by "diffiety" when treating a conceptual problem. For example, the question: what are the symmetries of partial differential equations, is to be formulated as

what are automorphisms of diffieties?

A little later we will see that there is no problem to answer it. But before that we must add some details into our picture.

First of all, we remark that starting from (10) one can produce many new labels, say, transforming dependent and independent variables or passing to the corresponding first-order system or extending it, etc. If $\mathscr{Y} \subset J^k(E, n)$ and $\mathscr{Y}' \subset J^{k'}(E', n)$ are two labels related to one another in this way, then $\mathscr{Y}_\infty = \mathscr{Y}'_\infty$.

This demonstrates clearly that $\mathcal{Y}$ and $\mathcal{Y}'$ are actually different "labels" of the same thing.

Now we have to interpret in terms of diffieties what are solutions of "partial differential equations". The concept of *integral submanifold* of a diffiety gives us the answer. More exactly, let $\mathcal{O}$ be a diffiety and Dim $\mathcal{O}=n$, i.e., dim $C_\theta=n$ for every $\theta\in\mathcal{O}$ where $\theta\mapsto C_\theta\subset T_\theta\mathcal{O}$ is the distribution with which $\mathcal{O}$ is supplied by definition. A submanifold $W\subset\mathcal{O}$ is called *integral* if $T_\theta W=C_\theta$ for every $\theta\in W$ (see Fig. 1). Evidently, dim $W=n$. It can be probed that every integral submanifold of the diffiety $\mathcal{O}=\mathcal{Y}_\infty$ is locally of the form $L_{(\infty)}$ where $L\subset E$ is a solution of $\mathcal{Y}$ in the usual sense of this word, i.e., the local equations (9) of L satisfy (10), by which $\mathcal{Y}$ is given locally (see Ref. [17]). This justifies our interpretation of solutions as integral submanifolds.

What was said before enables us to consider informally a diffiety as a shelf on which are stored all solutions of the corresponding differential equation. Also, it leads to an important generalization of the notion of solution for arbitrary nonlinear systems of partial differential equations (see Refs. [43,17,26,27,49,52,28,55]). We will discuss it below in connection with the quantization problem.

The following types of diffieties are of importance for us. Let N be a foliated manifold with n-dimensional leaves. For any $\theta\in N$ we define C_θ to be the tangent space to the leaf passing through θ. The manifold N equipped with the n-dimensional distribution $\theta\mapsto C_\theta$ is a diffiety. In fact, $N=\mathcal{Y}_\infty$ where $\mathcal{Y}$ is the system of partial differential equations composed of the Frobenius complete integrability conditions for the distribution $\{C_\theta\}$. Thus, foliated manifolds are diffieties. The diffiety dimension of such a diffiety is equal to the dimension of its leaves.

Also, every n-dimensional manifold, say M, can be regarded a foliated manifold consisting of only one leaf (in this case $C_\theta=T_\theta M$) and, therefore, as a diffiety. Evidently, in this case Dim $M=$ dim M.

Remark. There are two different natural ways to treat finite-dimensional mani-

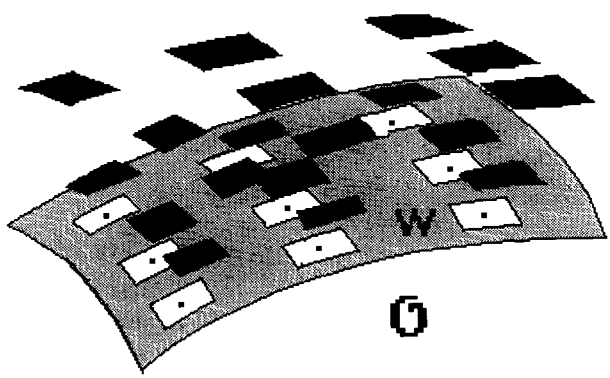

Fig. 1.

folds as diffieties. One of them was presented just above. We get the second one by supplying M with the zero-dimensional distribution $\theta \mapsto C_\theta = \{0\} \subset T_\theta M$. In this case Dim $M=0$. These two ways are, in a sense, dual to each other, and therefore, lead to a kind of duality in the theory of differential equations. Also, one can see that the traditional "differential" mathematics (Calculus, geometry, equations, etc.) viewed as a part of diffiety theory becomes conceptually closed only if the underlying manifolds are understood to be zero-dimensional diffieties. In other words, the standard "differential" mathematics forms the zero-dimensional part of diffiety theory. So, from this point of view it would be quite natural to suspect that the relevant mathematics necessary to quantize smoothly classical fields has not yet been discovered to a great extent.

8. What are symmetries of partial differential equations?

As we have already noted one can immediately answer this question by replacing the words "differential equations" in it with "diffieties".

Remembering that diffieties are manifolds equipped with a geometrical structure (the Cartan distribution) we see that their symmetries are to be diffeomorphisms which preserve this structure. More exactly we have:

Definition. A map $\Phi: \mathcal{O} \to \mathcal{O}$ is called a symmetry of the diffiety $\mathcal{O}$ if
 (i) Φ is a diffeomorphism,
 (ii) $d_\theta \Phi(C_\theta) = C_{\Phi(\theta)}$.

Here C_θ denotes, as before, the contact "plane" at θ and $d_\theta\Phi: T_\theta\mathcal{O} \to T_{\Phi(\theta)}\mathcal{O}$ denotes the differential of Φ at θ.

Specializing this definition to the case $\mathcal{O} = \mathcal{Y}_\infty$ we get the definition of symmetry for a concrete system of partial differential equations given by its label $\mathcal{Y}$, i.e. by (10).

Of course, symmetries of diffieties should constitute a special class of their morphisms. These morphisms are called smaps and their definition is as follows.

Definition. A map $\Phi: \mathcal{O}_1 \to \mathcal{O}_2$ of a diffiety $\mathcal{O}_1$ into a diffiety $\mathcal{O}_2$ is called a smap if
 (i) Φ is smooth, i.e., $f \circ \Phi \in C^\infty(\mathcal{O}_1)$ for every $f \in C^\infty(\mathcal{O}_2)$,
 (ii) $d_\theta\Phi(C_\theta) \subset C_{\Phi(\theta)}$ for every $\theta \in \mathcal{O}_1$.

It can be shown that smaps can be identified locally with differential operators, generally non-linear. Smaps are analogs of smooth maps of finite-dimensional manifolds. Also, the latter can be viewed as smaps of zero-dimensional diffieties.

Diffieties and smaps are objects and morphisms, respectively, of a category called the *category of differential equations*.

Now we have to introduce the infinitesimal version of diffiety symmetries. Let $\mathcal{O}$ be a diffiety. A vector field X on $\mathcal{O}$ is called a *trivial contact field* (on $\mathcal{O}$) if $X_\theta \in C_\theta$ for every $\theta \in \mathcal{O}$. Here X_θ denotes the vector of the field X assigned to θ.

Definition. A vector field X on $\mathcal{O}$ is called a *contact* field or a *$\mathscr{C}$-field* if the commutator $[X, Y]$ is a trivial contact field for every trivial contact field Y.

It results from the complete integrability of the contact structure on $\mathcal{O}$ that commutators of trivial contact vector fields are also trivial contact fields. In other words, the set $\mathscr{C}D(\mathcal{O})$ of all trivial contact fields on $\mathcal{O}$ is a Lie algebra.

Now it is easily seen from the definition and the Jacobi identity that $\mathscr{C}$-fields on $\mathcal{O}$ form a Lie algebra with respect to the standard commutator operation and that $\mathscr{C}D(\mathcal{O})$ is an ideal of it. We denote this algebra by $D_{\mathscr{C}}(\mathcal{O})$. The quotient Lie algebra

$$\mathrm{Sym}\ \mathcal{O} = D_{\mathscr{C}}(\mathcal{O}) / \mathscr{C}D(\mathcal{O})$$

is called the *symmetry algebra* of $\mathcal{O}$. Specializing this definition to the case $\mathcal{O} = \mathscr{Y}_\infty$ we come to the (higher-)symmetry algebra of a system of partial differential equations given by its label $\mathscr{Y}$ [or by (10)].

We refer the reader to Refs. [51,17] for the motivation of the above quotienting.

Elements of the algebra Sym $\mathscr{Y}$ are called (higher) *infinitesimal symmetries* of the system of partial differential equations given by the label $\mathscr{Y}$.

9. Infinitesimal symmetries of partial differential equations are secondary quantized vector fields

A description of the algebra Sym $\mathcal{O}$ in local coordinates will be needed to justify this assertion. We can restrict ourselves to the case $\mathcal{O} = \mathscr{Y}_\infty$ because every diffiety $\mathcal{O}$ is locally of this form.

First of all, we will consider the simplest situation when $\mathscr{Y}$ is the "empty" equation $0=0$. In this case $\mathscr{Y}_\infty = J^\infty(E, n)$ and we can use a local chart of the form $(x, u, ..., u^i_\sigma, ...)$ described above.

Coordinate expressions of trivial $\mathscr{C}$-fields in these coordinates look as

$$Y = \sum_{i=1}^{n} a_i D_i, \quad a_i \in \mathscr{F}(E, n),$$

where D_i is the total derivative operator (12).

Now, let $\varphi = (\varphi_1, ..., \varphi_m)$, $\varphi_i \in \mathscr{F}(E, n)$, be an $\mathbb{R}^m$-valued function on $J^\infty(E, n)$. This determines the so-called *evolutionary-derivation operator*

$$\mathfrak{E}_\varphi = \sum_{\sigma,i} D_\sigma(\varphi_i)\frac{\partial}{\partial u^i_\sigma}, \tag{15}$$

which is a vector field on $J^\infty(E, n)$. In fact, $\mathfrak{E}_\varphi$ is a $\mathscr{C}$-field and φ is called the *generating function* of it. The following result gives us the necessary local description of $\mathscr{C}$-fields.

Proposition. *Every $\mathscr{C}$-field X on $J^\infty(E, n)$ can be uniquely presented in the form*

$$X = \mathfrak{E}_\varphi + Y, \tag{16}$$

where Y is a trivial $\mathscr{C}$-field.

Also, we have

$$[\mathfrak{E}_\varphi, \mathfrak{E}_\psi] = \mathfrak{E}_{\{\varphi,\psi\}},$$

where

$$\{\varphi, \psi\} = \mathfrak{E}_\varphi(\psi) - \mathfrak{E}_\psi(\varphi). \tag{17}$$

This bracket operation supplies the linear space of all smooth $\mathbb{R}^m$-valued functions, defined on the considered local chart, with a Lie algebra structure. Then the above proposition shows this Lie algebra to be locally isomorphic with the Lie algebra Sym $J^\infty(E, n)$.

If $\chi \in \mathrm{Sym}\, J^\infty(E, n)$ and $\chi = \mathfrak{E}_\varphi \bmod \mathscr{C}D(J^\infty(E, n))$, φ is also called the *generating function of* χ (with respect to the chosen coordinate system).

Remark. The bracket (17) coincides with the standard Poisson bracket for functions $\varphi, \psi \in \mathscr{F}_1(E, n)$ supposing that $m = 1$ and φ, ψ do not depend on u.

Now we observe that every symmetry $\Phi: \mathscr{Y}_\infty \to \mathscr{Y}_\infty$ generates a map $\tilde{\Phi}$: Sol $\mathscr{Y} \to$ Sol $\mathscr{Y}$ of the "space" of all local solutions of $\mathscr{Y}$. In fact, it transforms an integral submanifold of $\mathscr{Y}_\infty$ into another one as is clearly seen from the definitions. Therefore, identifying integral submanifolds of $\mathscr{Y}_\infty$ with local solutions of $\mathscr{Y}$ as was explained before we get the map $\tilde{\Phi}$.

When $\mathscr{Y}$ is the "empty" equation $0 = 0$, then $\mathscr{Y}_\infty = J^\infty(E, n)$ and the set Sol $\mathscr{Y}$ is canonically identified with the set of all n-dimensional submanifolds of E. Therefore, every symmetry $\Phi: J^\infty(E, n) \to J^\infty(E, n)$ generates a transformation $\tilde{\Phi}$ of the "space" of all n-dimensional submanifolds of E.

By the same reasoning every infinitesimal symmetry of $\mathscr{Y}_\infty$, i.e. every $\mathscr{C}$-field, say X, on $\mathscr{Y}_\infty$ generates a (virtual) flow on the "space" Sol $\mathscr{Y}$. The velocity field of this flow is given by the formula

$$\frac{\partial u^i}{\partial t} = \varphi_i(x, u, \ldots, u^i_\sigma, \ldots), \quad i = 1, \ldots, m, \tag{18}$$

in which t is a new independent variable (the "evolution time") and $X=\mathfrak{E}_\varphi+Y$, $\varphi=(\varphi_1, ..., \varphi_m)$, according to the above proposition.

It is seen from (18) that the flow on Sol $\mathscr{Y}$ generated by X does not depend on its trivial part Y. In other words, $\mathscr{C}$-fields belonging to the same coset modulo $\mathscr{C}D(\mathscr{Y}_\infty)$ generate the same flow on Sol $\mathscr{Y}$. This is why the Lie algebra $D_{\mathscr{C}}(\mathscr{O})$ of all contact fields is to be quotiented by trivial ones in order to obtain true symmetries. The meaning of generating functions becomes clear from (18).

Now we note that the generating function occuring in (18) is not arbitrary unless $\mathscr{Y}_\infty=J^\infty(E, n)$. In fact, it must satisfy the following equation, supposing that $\mathscr{Y}$ is given by (10):

$$\bar{l}_F\bar{\varphi}=0 .$$

Here $F=(F_1, ..., F_l)$ and

$$l_F=\begin{pmatrix} \sum_\sigma \dfrac{\partial F_1}{\partial u_\sigma^1} D_\sigma & \cdots & \sum_\sigma \dfrac{\partial F_1}{\partial u_\sigma^m} D_\sigma \\ \vdots & & \vdots \\ \sum_\sigma \dfrac{\partial F_l}{\partial u_\sigma^1} D_\sigma & \cdots & \sum_\sigma \dfrac{\partial F_l}{\partial u_\sigma^m} D_\sigma \end{pmatrix}$$

is an $(l\times m)$ matrix differential operator on $J^\infty(E, n)$ and bars over l_F and φ indicate the restrictions to $\mathscr{Y}_\infty$.

It is natural now to ask: what do the notions just introduced mean when applied to zero-dimensional diffieties, i.e. to usual manifolds (see the end of section 7). In this case the contact distribution is zero dimensional and, therefore, $\mathscr{C}D(M)=0$ for a manifold M viewed as a zero-dimensional diffiety. By the same reasoning every vector field on M is contact and we see that Sym $M=D(M)$, where $D(M)$ stands for the Lie algebra of all vector fields on M.

Since $n=0$ in the situation in question, every local chart on M, say $u^1, ..., u^m$, can be regarded as a divided one. This shows that the standard coordinate expression $X=\sum_i \varphi_i(u)\ \partial/\partial u^i$, $u=(u^1, ..., u^m)$, for a vector field X on M is a particular case of (15). Namely, we have $X=\mathfrak{E}_\varphi$, for $\varphi=(\varphi_1(u), ..., \varphi_m(u))$, and the system (18) looks in this case as

$$\frac{\mathrm{d}u^i}{\mathrm{d}t}=\varphi_i(u^1, ..., u^m) , \quad i=1, ..., m . \tag{19}$$

But in (19) we recognize the ordinary differential equations of characteristics for the first-order partial differential operator $X=\sum\varphi_i\partial/\partial u^i$. Now the analogy $X\mapsto\mathfrak{E}_\varphi$, (19) $\mapsto$ (18) motivates the following principal statement:

If $\varphi=(\varphi_1, ..., \varphi_m)$ is the generating function of a symmetry $\chi\in$ Sym $\mathscr{Y}_\infty$, then the *system (18) of partial differential equations can be regarded naturally as the characteristic system corresponding to the operator χ.*

In virtue of (3), this gives the desired solution of the main problem (4) for first-order operators:

Symmetries of partial differential equations are first-order secondary quantized differential operators.

This is our starting point when we look for secondary quantized differential operators of higher orders. Before, however, we will discuss some perspectives of the symmetry theory for partial differential equations, because of its potential importance for the future secondary calculus.

10. Digression: on symmetries of partial differential equations

Besides the just mentioned aim, in the first part of this section we will collect some brief historical remarks in order to illustrate the previous discussion and, also, for some terminological reasons.

The symmetry theory for differential equations was founded by Lie, who created both the conceptual structure of this theory and the main technical tools of it (see Ref. [23]). Unfortunately, only the two simplest, in a sense, aspects of his work, namely Lie groups and Lie algebras, were mainly assimilated by the mathematical community, and then developed in thousands of works which have nothing in common with differential equations. As we have already mentioned, the symmetries of a differential equation by Lie are transformations of dependent and independent variables, or the first-order contact transformations, discovered by Lie himself, which leave invariant the chosen label of this equation. We remark that the so-defined symmetry group or algebra can actually depend on the choice of a label of the equation in question. Also, generating functions of these classical infinitesimal symmetries can depend only on derivatives not higher than first order. This makes clear the interrelations between classical and modern symmetry theories.

The first systematic attempts to apply Lie's theory to the mechanics of continuous media were made by Ovsiannikov and his collaborators about 70–80 years after Lie's original work (see Ref. [34]). These authors merely elaborated some technical aspects of Lie's theory, while its conceptual content remained untouched. Probably, Ovsiannikov was the first who recognized Lie's theory behind many special methods and tricks in use in mechanics. The so-called dimensional analysis by Sedov and Birkhoff (see Refs. [37], [4]) is maybe the most remarkable example of this kind.

It looks surprising that the first attempts to generalize the classical notion of symmetry were made almost immediately after Lie's work, first by Bäcklund and then by Noether. For example, one can find a coordinate-wise definition of $\mathscr{C}$-fields on $J^\infty(\mathbb{R}^{n+1}, n)$ in Ref. [3] by Bäcklund. Of course, the mentioned works of Bäcklund and Noether could not be well based at that time, because of lack of experience in working with infinite-dimensional manifolds. For example, Bäcklund considered as integrable all vector fields on infinite jets. Moreover, one can find the same deficiency in some recent works (for instance, Ref. [2]).

New times for the symmetry theory arrived with the discovery of non-linear equations integrable via the so-called inverse scattering transform method. The first examples of non-classical ("higher") symmetries were found at that time. For instance, it turned out that the higher analogs of the Korteweg–de Vries equation are, in fact, its non-classical symmetries.

Apparently, the rigorous non-classical symmetry theory starts with the work by Kupershmidt [19] in which all $\mathscr{C}$-fields on $J^\infty\pi$ were completely described. Then the author introduced all necessary basic notions, constructions and formulas of this theory, some of which were sketched above [44,46,47]. Later on, some of them were repeated by Ibragimov [12,13] and used by Olver in his textbook [33].

We call these new non-classical symmetries "higher" to emphasize that their generating functions can depend on derivatives of arbitrary order, unlike the classical ones, which can depend only on derivatives of order not greater than one. In the current literature the terms "generalized symmetries" (for instance, Ref. [33]) and "Lie–Bäcklund transformations" (for instance, Ref. [13]) are also used in the same sense.

The sketched symmetry theory is presented "in action" in Ref. [56], where attention is paid to the relevant computational aspects including the problem of computerization.

It seems that foundations of the "higher" symmetry theory as well as the corresponding computational algorithms are now well established. So, the main general problem in this field is to enlarge the area of possible applications developing and elaborating the underlying techniques. Another very interesting problem here is: what are the obstacles for partial differential equations to be symmetric? It is remarkable that this problem admits a natural solution in terms of secondary differential calculus. Unfortunately, we cannot touch upon this topic here.

On the other hand, one can imagine naturally a generalization of this theory in which generating functions of symmetries could depend not only on derivatives of arbitrary order but also on "non-local" variables such as $\int f \cdot \mathrm{d}x, f \in \mathscr{F}$. In fact, there exist a number of problems suggesting such a generalization, and the first few steps in this direction have already been taken [16,22,14,32,7,5].

We can apply once again the philosophy of section 4 in searching the desired concept of non-local symmetry. This means answering the question "what are differential equations" in a different manner than in section 4. Below, we will indicate very briefly how this can be done (see Ref. [16] for more details and motivations).

Let $\mathcal{O}$, $\mathcal{O}'$ be diffieties, Dim $\mathcal{O}$ = Dim $\mathcal{O}'$. A smap Φ: $\mathcal{O} \to \mathcal{O}'$ is said to be a *covering* of $\mathcal{O}$ if $\mathrm{d}_\theta\Phi(C_\theta) = C_{\Phi(\theta)}$, $\forall\theta \in \mathcal{O}'$. A symmetry $\chi \in \mathrm{Sym}\ \mathcal{O}'$ is said to be a *non-local (infinitesimal) symmetry*, or, more exactly, a Φ-symmetry of $\mathcal{O}$. We stress that non-local symmetries of $\mathcal{O}$ are pairs of the form (Φ, χ), where Φ is a covering of $\mathcal{O}$ and χ is a "usual" symmetry of the covering diffiety $\mathcal{O}'$. Roughly speaking, the reason to call the so-defined symmetries non-local is that functions on $\mathcal{O}'$ (in particular, the generating ones) are seen by an "observer" on $\mathcal{O}$ as depending on some new ("non-local") variables compared to those on $\mathcal{O}$.

Example. Consider the Burgers equation $\mathcal{Y} = \{u_t = uu_x + u_{xx}\}$ and the heat equation $\tilde{\mathcal{Y}} = \{v_t = v_{xx}\}$. The Cole–Hopf substitution $u = 2v_x/v$ connecting these equations is, in fact, the label of a covering Φ: $\tilde{\mathcal{Y}}_\infty \to \mathcal{Y}_\infty$. Therefore, higher symmetries of the heat equation can be considered as non-local symmetries of the Burgers equation. The heat equation is linear and so every solution $a(x, t)$ of it can be treated also as a symmetry of it with generating function $a(x, t)$. On the other hand, the generating function of the corresponding non-local symmetry of the Burgers equation is

$$(2a_x - au)\exp\left(-\frac{1}{2}\int u\,\mathrm{d}x\right).$$

It involves a non-local variable, namely, $\int u\,\mathrm{d}x$.

The so-defined non-local symmetries can be used in the standard applications exactly in the same way as the classical or higher ones. However, the fact that different non-local symmetries live, generally, on different coverings lead to some non-standard and surprising features of non-local theory. One of them is the following.

Evidently, all Φ-symmetries of $\mathcal{O}$ constitute for a fixed Φ: $\mathcal{O}' \to \mathcal{O}$ a Lie algebra which coincides with Sym $\mathcal{O}'$. But, at first glance, it seems to be absurd to look for the commutator of two non-local symmetries defined on two different coverings. However, it turns out to be possible to find the desired commutator on a suitable third covering. Therefore, in order to organize all non-local symmetries of $\mathcal{O}$ into something like a Lie algebra one must take into consideration simultaneously all coverings of $\mathcal{O}$.

All coverings of a given diffiety constitute in a natural way a category which we have called a *cobweb* (see Ref. [16]). Now the philosophy of section 4 leads us to answer the question what are differential equations by saying that *they are cobwebs*. This answer implies many important consequences for secondary calculus. But at the moment this is only a beautiful perspective to be explored systematically.

We remark also that, while diffieties are analogs of affine varieties of algebraic geometry, cobwebs are analogs of fields of rational functions on them.

11. Secondary ("quantized") functions

It seems natural to define higher-order secondary quantized differential operators as compositions of first-order ones. But going this way we meet immediately the following difficulty.

Remember that first-order secondary differential operators are elements of the Lie algebra Sym $\mathcal{O}$. On the other hand, the latter are not proper differential operators but cosets (equivalence classes) of them. So the question arises: how to compose these cosets? We leave to the interested reader the task of verifying that a direct attack to this problem fails.

Another aspect of the problem can be extracted from a similar question: on what kind of objects do secondary differential operators act? No doubt, secondary differential operators should be proper operators, i.e. act on some kind of objects. The usual functions cannot be taken as such. One can see this by trying to define an action of the algebra Sym $\mathcal{O}$ on $C^\infty(\mathcal{O})$. The only natural way to do this is to put $\chi(f)=X(f)$ for $\chi\in\mathrm{Sym}\,\mathcal{O}$, $X\in D_{\mathscr{C}}(\mathcal{O})$, $\chi=X \bmod \mathscr{C}D(\mathcal{O})$ and $f\in C^\infty(\mathcal{O})$. But this definition is clearly not correct. Namely, if $X_1, X_2\in D_{\mathscr{C}}(\mathcal{O})$ and $X_1\equiv X_2 \bmod \mathscr{C}D(\mathcal{O})$, then, generally, $X_1(f)\neq X_2(f)$ if $X_1\neq X_2$.

However, it is clear that, linguistically, secondary operators should act on secondary functions and we shift this question by asking what they are. The following analogy will help us to answer.

Let $\Lambda^i(M)$ denote the space of ith-degree differential forms on the manifold M. The map

$$C^\infty(M)\xrightarrow{\mathrm{d}}\Lambda^1(M) \tag{20}$$

(the standard differential) provides extremal problems on smooth functions on M with the "universal solution". Treating smooth manifolds as zero-dimensional diffieties we see that the analog of (20) should be a map which provides variational problems for multiple integrals with the universal solution. But this is the well-known Euler–Lagrange map:

$$\boxed{\begin{matrix}\text{variational functionals,}\\ \text{or "actions"}\end{matrix}} \xrightarrow{\mathscr{E}} \boxed{\begin{matrix}\text{differential}\\ \text{operators}\end{matrix}} \tag{21}$$

i.e. $\mathscr{E}$ associates with an "action" $\int_\Omega L\,\mathrm{d}x_1\cdots\mathrm{d}x_n$ the left-hand side of the corresponding Euler–Lagrange equation. Therefore, this analogy between (20) and (21) suggests to adopt "actions" as secondary (or "quantized") functions. This idea is to be corrected because "actions", as understood in the standard way, contain a detail, parasitic for our aim, to be eliminated. This detail is the exact reference to the domain of integration Ω. So, our next problem is to find a meaning for hieroglyphs of the form $\int L\, dx_1 \cdots dx_n$ (without "Ω"!). We will solve it by interpreting them as some kind of cohomology classes (for more motivations and details see Ref. [9]). But before that we need some preliminaries.

Let $\mathcal{O}$ be a diffiety, Dim $\mathcal{O}=n$ and let $\Lambda^i(\mathcal{O})$ denote the $C^\infty(\mathcal{O})$-module of all ith-degree differential forms on $\mathcal{O}$. We denote by $\mathscr{C}\Lambda^i(\mathcal{O})$ the submodule of $\Lambda^i(\mathcal{O})$ consisting of forms whose restrictions on the contact distribution of $\mathcal{O}$ vanish. In other words,

$$\mathscr{C}\Lambda^i(\mathcal{O}) \ni \omega \Leftrightarrow \omega(Y_1, \ldots, Y_1)=0\,, \quad \exists Y_1, \ldots, Y_i \in \mathscr{C}D(\mathcal{O})\,.$$

We put

$$\bar{\Lambda}^i(\mathcal{O})=\Lambda^i(\mathcal{O})/\mathscr{C}\Lambda^i(\mathcal{O})\,.$$

Elements of $\bar{\Lambda}^i(\mathcal{O})$ are called *horizontal differential forms* on $\mathcal{O}$. Evidently, $\bar{\Lambda}^i(\mathcal{O})=0$ if $i>n$.

It is easy to see that $\mathrm{d}(\mathscr{C}\Lambda^i(\mathcal{O})) \subset \mathscr{C}\Lambda^{i+1}(\mathcal{O})$ and, therefore, the standard differential $\mathrm{d}: \Lambda^i(\mathcal{O}) \to \Lambda^{i+1}(\mathcal{O})$ induces the *horizontal differential*

$$\bar{\mathrm{d}}: \bar{\Lambda}^i(\mathcal{O}) \to \bar{\Lambda}^{i+1}(\mathcal{O})\,.$$

Of course, $\bar{\mathrm{d}}^2=0$ and this enables us to introduce the horizontal de Rham complex of $\mathcal{O}$:

$$0 \to \bar{\Lambda}^0(\mathcal{O})=C^\infty(\mathcal{O}) \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^1(\mathcal{O}) \xrightarrow{\bar{\mathrm{d}}} \cdots \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^n(\mathcal{O}) \to 0\,.$$

Cohomologies of this complex are called *horizontal de Rham cohomologies of* $\mathcal{O}$. They are denoted by $\bar{H}^i(\mathcal{O})$, $i=0, 1, \ldots, n$.

Finally, we accept the following basic interpretation:

Secondary (or "quantized") functions on $\mathcal{O}$ are elements of the cohomology group $\bar{H}^n(\mathcal{O})$.

In other words, we consider the cohomology group $\bar{H}^n(\mathcal{O})$, Dim $\mathcal{O}=n$, to be the analog of the smooth function algebras in Secondary Calculus.

To justify the choice made we will describe in coordinates the "horizontal" constructions, just given, for $\mathcal{O}=J^\infty(E, n)$. First of all, we observe that the coset of a differential form $\omega \in \Lambda^i(J^\infty(E, n))$ modulo $\mathscr{C}\Lambda^i(J^\infty(E, n))$ contains only one element of the form

$$\rho = \sum_{1<k_1<\cdots<k_i<n} a_{k_1\cdots k_i}(x, u, \ldots, u^j_\sigma, \ldots)\, \mathrm{d}x_{k_1} \wedge \cdots \wedge \mathrm{d}x_{k_i}\,,$$

where $a_{k_1\cdots k_i} \in C^\infty(J^\infty(E, n))$. The characteristic feature of such forms is that the differentials $\mathrm{d}u^j, \ldots, \mathrm{d}u^j_\sigma, \ldots$ do not enter into their coordinate expressions. Therefore, the module $\bar{\Lambda}^i(J^\infty(E, n))$ can be identified locally with the module of forms of this type.

Under the identification made, the horizontal de Rham differential $\bar{\mathrm{d}}$ looks as

$$\bar{\mathrm{d}}\rho = \sum_{s,k_1,\ldots,k_i} D_s(a_{k_1\cdots k_i})\, \mathrm{d}x_s \wedge \mathrm{d}x_{k_1} \wedge \cdots \wedge \mathrm{d}x_{k_i}\,.$$

In particular, every horizontal $(n-1)$-form can be represented uniquely as

$$\rho = \sum_i (-1)^{i-1} a_i\, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\,, \quad a_i \in C^\infty(J^\infty(E, n))\,,$$

and

$$\bar{\mathrm{d}}\rho = \mathrm{div}(A)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n\,,$$

where $A = (a_1, \ldots, a_n)$ and $\mathrm{div}\, A = \sum_i D_i(a_i)$. Also, horizontal n-forms look as

$$L(x, u, \ldots, u^i, \ldots)\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\,, \quad L \in C^\infty(J^\infty(E, n))\,,$$

and one can recognize lagrangian densities in them. So, we see that the horizontal cohomology $\bar{H}^n(J^\infty(E, n))$ can be identified locally with the linear space of equivalence classes of lagrangian densities on $J^\infty(E, n)$ with respect to the following relation:

$$L_1 \sim L_2 \quad \Leftrightarrow \quad L_1 - L_2 = \mathrm{div}\, A \ \text{ for some } A\,.$$

On the other hand, actions $\int_\Omega L_i\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$, $i = 1, 2$ are equivalent in the sense that they lead to identical Euler–Lagrange equations iff $L_1 \sim L_2$. This is independent of the choice of Ω. For these reasons it is natural to identify hieroglyphs $\int L\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$ with n-dimensional horizontal cohomology classes.

We conclude this section noting that similar reasonings are valid for arbitrary diffieties as well.

12. Higher-order scalar secondary ("quantized") differential operators

First of all, we must justify the above adopted definition of secondary functions, demonstrating that, indeed, first-order differential operators act naturally on them. In other words, we must look for a natural action of the algebra Sym $\mathcal{O}$ on the space $\bar{H}^n(\mathcal{O})$. This, however, can be done straightforwardly.

First, note that, if $X \in D_{\mathscr{C}}(\mathcal{O})$, $\omega \in \mathscr{C}\Lambda^i(\mathcal{O})$ and L_X denotes the Lie derivative along X, then $L_X(\omega) \in \mathscr{C}\Lambda^i(\mathcal{O})$ as results from definitions and the fact that

$$\mathscr{C}\Lambda^i(\mathscr{O}) = \mathscr{C}\Lambda^1(\mathscr{O}) \wedge \Lambda^{i-1}(\mathscr{O}) , \quad i>1 .$$

This allows us to define the Lie derivative on horizontal forms by passing to quotients:

$$L_X : \bar{\Lambda}^i(\mathscr{O}) \to \bar{\Lambda}^i(\mathscr{O}) .$$

Next, let

$$\chi \in \mathrm{Sym}\, \mathscr{O} , \quad \chi = X \bmod \mathscr{C}D(\mathscr{O}) \quad \text{for } X \in D_{\mathscr{C}}(\mathscr{O}) ,$$

$$\bar{\theta} \in \bar{H}^n(\mathscr{O}) , \quad \theta = \omega \bmod \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O}) \quad \text{for } \omega \in \bar{\Lambda}^n(\mathscr{O}) .$$

We define now the action of χ on θ by putting

$$\chi(\theta) := \{L_X(\omega) \bmod \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})\} \in \bar{H}^n(\mathscr{O}) ,$$

where L_X denotes the Lie derivative along X. The correctness of this definition follows directly from the following two facts:

(i) $L_Y(\omega) \in \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})$ if $\omega \in \bar{\Lambda}^n(\mathscr{O})$ and $Y \in \mathscr{C}D(\mathscr{O})$,

(ii) $L_X \circ \bar{\mathrm{d}} = \bar{\mathrm{d}} \circ L_X$.

They both are direct consequences of definitions.

Now we see that the above definition of secondary functions correlates nicely with other "secondary" constructions and, therefore, can serve as an example in proceeding to more complicated "secondary" notions. For example, let us observe that we have succeeded to define a correct action of one quotient [namely, $\mathrm{Sym}\, \mathscr{O} = D_{\mathscr{C}}(\mathscr{O}) / \mathscr{C}D(\mathscr{O})$] on another [namely, $\bar{H}^n(\mathscr{O}) = \bar{\Lambda}^n(\mathscr{O}) / \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})$] owing to:

1. $D_{\mathscr{C}}(\mathscr{O})$ consists of first-order differential operators which act on the $C^\infty(\mathscr{O})$-module $\bar{\Lambda}^n(\mathscr{O})$ leaving $\bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})$ invariant;

2. the images of $\bar{\Lambda}^n(\mathscr{O})$ under the action of first-order operators belonging to $\mathscr{C}D(\mathscr{O})$ are contained in $\bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})$.

We will get the necessary generalization to higher-order secondary differential operators simply by replacing the words "first-order" by "kth-order" in 1 and 2 above. More exactly, let $\mathrm{Diff}_k(\bar{\Lambda}^n(\mathscr{O}))$ denote the $C^\infty(\mathscr{O})$-module of all ("usual") differential operators of order $\leqslant k$ acting on $\bar{\Lambda}^n(\mathscr{O})$ and put

$$\overline{\mathrm{Diff}}_k(\mathscr{O}) = \{\Delta \in \mathrm{Diff}_k(\bar{\Lambda}^n(\mathscr{O})) \mid \Delta(\bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})) \subset \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})\} ,$$

$$\underline{\mathrm{Diff}}_k(\mathscr{O}) = \{\Delta \in \mathrm{Diff}_k(\bar{\Lambda}^n(\mathscr{O})) \mid \Delta(\bar{\Lambda}^n(\mathscr{O})) \subset \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})\} .$$

Then the space of all *scalar secondary ("quantized") operators* of order $\leqslant k$ on $\mathscr{O}$ is defined to be the quotient

$$\mathfrak{Diff}(\mathscr{O} = \overline{\mathrm{Diff}}_k(\mathscr{O}) / \underline{\mathrm{Diff}}_k(\mathscr{O}) . \tag{22}$$

Of course, every *secondary operator* $\Delta \in \mathfrak{Diff}_k(\mathscr{O})$ can be understood as an actual operator

$$\Delta : \bar{H}^n(\mathscr{O}) \to \bar{H}^n(\mathscr{O})$$

acting on secondary functions. In fact, if

$$\Delta = \delta \bmod \underline{\mathrm{Diff}}_k(\mathscr{O}) \quad \text{for} \quad \delta \in \overline{\mathrm{Diff}_k}(\mathscr{O}) ,$$

$$\Theta = \omega \bmod \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O}) \quad \text{for} \quad \omega \in \bar{\Lambda}^n(\mathscr{O}) ,$$

then the horizontal cohomology class

$$\Delta(\Theta) := \{\delta(\omega) \bmod \bar{\mathrm{d}}\bar{\Lambda}^{n-1}(\mathscr{O})\}$$

is well defined, i.e., does not depend on the choice of the representatives δ and ω.

For $\mathscr{O} = J^\infty(E, n)$ the so-defined secondary differential operators admit the following coordinate description. Operators of the form

$$\sum_{s=1}^{k} \sum_{\substack{i_1 \cdots i_s \\ \sigma_1 \cdots \sigma_s}} a^{i_1 \cdots i_s}_{\sigma_1 \cdots \sigma_s} \frac{\partial^s}{\partial p^{i_1}_{\sigma_1} \cdots \partial p^{i_s}_{\sigma_s}} + \text{const.} ,$$

where $\sigma_1, \ldots, \sigma_s$ are multi-indices, are called *vertical* (with respect to the chosen coordinate system). Then it can be proved that every coset

$$\Delta = \{\delta \bmod \underline{\mathrm{Diff}}_k(J^\infty(E, n))\} \in \mathfrak{Diff}(J^\infty(E, n))$$

for $\delta \in \mathrm{Diff}_k(J^\infty(E, n))$, contains only one vertical operator. So, the quotient (22) representing secondary differential operators can be identified locally with the set of all *vertical secondary operators*. These last operators of order $\leqslant k$ can be presented in the form

$$\mathfrak{E}_V := \sum_{i=1}^{m} \sum_{\sigma} \mathscr{L}_\sigma(V^i) \circ \partial / \partial p^i_\sigma ,$$

where $V = (V^1, V^2, \ldots, V^m)$, $V^i \in \mathrm{Diff}_{k-1}\mathscr{F}(E, n)$, are arbitrary vertical operators and

$$\mathscr{L}_\sigma(V^i) = [D_{i_1}, \ldots, [D_{i_n}, V^i] \cdots] ,$$

where $\sigma = (i_1, \ldots, i_n)$. The generating operator V is the higher-order analog of the generating functions for evolutionary derivations but, unlike the latter, it is not defined uniquely if $k > 1$.

Secondary operators of order > 1 are not reduced to compositions of first-order secondary operators. This fact is instructive in connection with the discussion at the beginning of section 11.

We note, also, that an explicit description of secondary differential operators on arbitrary diffieties is a much more difficult problem.

Further details, results and alternative views concerning secondary differential operators can be found in Ref. [9].

Finally, turning back to question (4) we can exhibit the simplest kth order linear secondary (quantized) differential equations as

$$\Delta(H)=0\,,\qquad \Delta\in\mathfrak{Diff}_{\mathfrak{t}}(\mathcal{O})\,,\quad H\in\bar{H}^n(\mathcal{O})\,.$$

It must be emphasized, however, that these equations form only a very special class of secondary differential equations. For instance, differentials $\mathrm{d}_1^{p,q}=\mathrm{d}_1^{p,q}(\mathcal{O})$ of the $\mathscr{C}$-spectral sequence (see sections 13 and 14) give us other examples of secondary quantized differential operators and, therefore, secondary differential equations. One of them looks as

$$\mathscr{E}\left(\int L\,\mathrm{d}x\right)=0\,,$$

where $\int L\,\mathrm{d}x\in\bar{H}^n(\mathcal{O})$ and $\mathscr{E}$ is the Euler operator assigning to an action $\int L\,\mathrm{d}x\in\bar{H}^n(\mathcal{O})$ the corresponding Euler–Lagrange equation. This is due to the fact that $\mathscr{E}=\mathrm{d}_1^{0,n}$ (see section 14). We note also that operators $\mathrm{d}_1^{p,q}$ are of finite order, say $\pi(k)$, when being restricted to elements of the kth filtration, but $\pi(k)\to\infty$ when $k\to\infty$. For example, $\pi(k)=2k$ for the operator $\mathrm{d}_1^{0,n}=\mathscr{E}$.

13. Secondary ("quantized") differential forms. $\mathscr{C}$-spectral sequences

In this section we will consider another aspect of secondary calculus, namely, secondary ("quantized") differential forms. What are they? This is a more difficult question than the one about secondary differential operators we have already discussed. For this and other reasons we will omit here the preliminary motivations showing, as before, how to arrive at exact definitions. However, some "a posteriori" justifications will be given.

Let $\mathcal{O}$ be a diffiety. Adopting the notations of section 12 we consider the algebra

$$\Lambda^*(\mathcal{O})=\sum_{i\geqslant 0}\Lambda^i(\mathcal{O})$$

of all differential forms on $\mathcal{O}$ and its ideal

$$\mathscr{C}\Lambda^*(\mathcal{O})=\sum_{i\geqslant 0}\mathscr{C}\Lambda^i(\mathcal{O})\,.$$

Denote by $\mathscr{C}^k\Lambda^*(\mathcal{O})$ the kth power of this ideal, which is, by definition, the linear subspace of $\Lambda^*(\mathcal{O})$ generated by all products of the form $\omega_1\wedge\omega_2\wedge\cdots\wedge\omega_k$, $\omega_i\in\mathscr{C}\Lambda^*(\mathcal{O})$. The ideal $\mathscr{C}\Lambda^*(\mathcal{O})$ and, therefore, all ideals $\mathscr{C}^k\Lambda^*(\mathcal{O})$ are stable with respect to the exterior differential d. So, we get the following filtration:

$$\Lambda^*(\mathcal{O})\supset\mathscr{C}\Lambda^*(\mathcal{O})\supset\mathscr{C}^2\Lambda^*(\mathcal{O})\supset\cdots\supset\mathscr{C}^k\Lambda^*(\mathcal{O})\supset\cdots\tag{23}$$

of the de Rham complex of $\mathcal{O}$ by its subcomplexes $\{\mathscr{C}^k\Lambda^*(\mathcal{O}),\,\mathrm{d}\}$. The ideals

$\mathscr{C}^k\Lambda^*(\mathcal{O})$ are naturally graded,

$$\mathscr{C}^k\Lambda^*(\mathcal{O}) = \sum_{s\geq 0} \mathscr{C}^k\Lambda^{k+s}(\mathcal{O}) ,$$

where

$$\mathscr{C}^k\Lambda^{k+s}(\mathcal{O}) = \mathscr{C}^k\Lambda^*(\mathcal{O}) \cap \Lambda^{k+s}(\mathcal{O}) .$$

Now we will achieve our goal by making use of standard spectral sequence arguments applied to the filtration (23) (see, for instance, Ref. [31] for the general theory). We start with the quotients

$$E_0^{p,q}(\mathcal{O}) := \mathscr{C}^p\Lambda^{p+q}(\mathcal{O}) / \mathscr{C}^{p+1}\Lambda^{p+q}(\mathcal{O})$$

connected in succession with the differentials

$$\mathrm{d}_0^{p,q} : E_0^{p,q}(\mathcal{O}) \to E_0^{p,q+1}(\mathcal{O}) ,$$

which are natural quotients of the exterior differential d. They all constitute a complex, namely,

$$E_0(\mathcal{O}) = \sum_{p,q} E_0^{p,q}(\mathcal{O}) , \quad \mathrm{d}_0 = \sum_{p,q} \mathrm{d}_0^{p,q} , \quad \mathrm{d}_0 : E_0(\mathcal{O}) \to E_0(\mathcal{O}) ,$$

which is called the zero term of the spectral sequence corresponding to the filtration (23). Its *first term* is then defined to be the cohomology of the zero term. More exactly, we put

$$E_1^{p,q}(\mathcal{O}) := \ker \mathrm{d}_0^{p,q} / \mathrm{im}\ \mathrm{d}_0^{p,q-1}$$

and note that the exterior differential d generates by passing to quotients the differentials

$$\mathrm{d}_1^{p,q} : E_1^{p,q}(\mathcal{O}) \to E_1^{p+1,q}(\mathcal{O}) .$$

The first term is now defined to be the complex

$$E_1(\mathcal{O}) = \sum_{p,q} E_1^{p,q}(\mathcal{O}) , \quad \mathrm{d}_1 = \sum_{p,q} \mathrm{d}_1^{p,q} , \quad \mathrm{d}_1 : E_1(\mathcal{O}) \to E_1(\mathcal{O}) .$$

Continuing this procedure one can define the *rth term*

$$E_r(\mathcal{O}) = \sum_{p,q} E_r^{p,q}(\mathcal{O}) , \quad \mathrm{d}_r = \sum_{p,q} \mathrm{d}_r^{p,q} ,$$

$$\mathrm{d}_r^{p,q} : E_r^{p,q}(\mathcal{O}) \to E_r^{p+r,q-r+1}(\mathcal{O}) ,$$

as cohomology of the $(r-1)$th term.

The so defined system $\{E_r(\mathcal{O}), \mathrm{d}_r\}$ of complexes is called the *$\mathscr{C}$-spectral sequence* of the diffiety $\mathcal{O}$. It is often convenient to display the terms of a spectral sequence in a diagram as shown on Fig. 2. For example, if Dim $\mathcal{O} = n$, then the

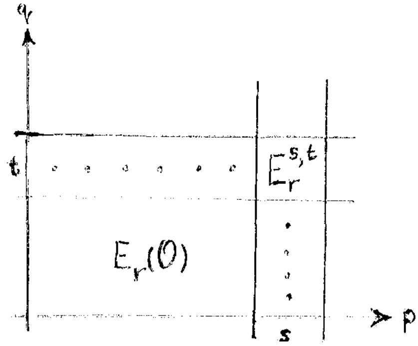

Fig. 2.

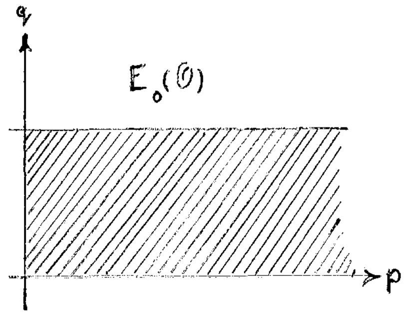

Fig. 3.

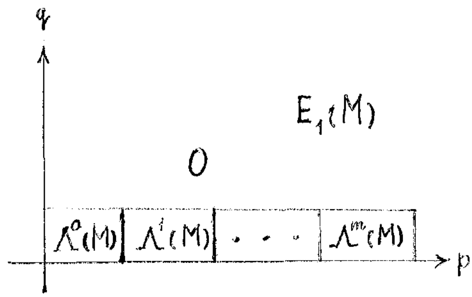

Fig. 4.

structure of the zero term of the $\mathscr{C}$-spectral sequence of $\mathcal{O}$ is illustrated by the diagram of Fig. 3; all its non-trivial terms $E_0^{p,q}(\mathcal{O})$ are situated in the shaded region. The same picture is obtained for all $E_r(\mathcal{O})$, $r>0$, as is easily deduced from the definitions.

Definition. Elements of $E_1(\mathcal{O})$ are called *secondary ("quantized") differential forms* on the diffiety $\mathcal{O}$.

Some reasons in favor of this interpretation are as follows. Let M be a finite-dimensional manifold considered as a zero-dimensional diffiety (see section 7). Then the diagram for the first term of its $\mathscr{C}$-spectral sequence looks as shown in Fig. 4. Moreover, $d_1^{p,0} = d$. In other words, we see that the de Rham complex of M coincides with the (generally) non-trivial part of the first term of its $\mathscr{C}$-spectral sequence.

We can observe that standard constructions and formulae connecting "usual" vector fields and differential forms are also valid for their secondary ("quantized") analogs. For instance, the insertion operator of secondary vector fields ("symmetries") into secondary differential forms as well as the corresponding Lie derivatives are well defined. Moreover, they are connected by means of the secondary analog of the infinitesimal Stokes formula

$$L_X = i_X \circ d + d \circ i_X ,$$

in which the exterior differential d is to be replaced by its secondary analog, i.e. by $d_1(\mathcal{O})$.

Finally, we remark that secondary differential forms are bigraded objects unlike the "usual" ones which are only monograded. The reason is clearly seen from the above diagrams. This is an illustration of the fact that secondary objects are richer and more complicated structures than their "primary" analogs. The same idea can be expressed alternatively by saying that the "usual" (or "primary") mathematical objects are degenerate forms of the secondary ones. This statement can be also viewed as the following mathematical paraphrase by the Bohr correspondence principle:

$$\boxed{\text{Secondary Calculus}} \xrightarrow{\text{Dim}\to 0} \boxed{\text{Calculus}}$$

The cobweb theory (see the end of section 10) allows us to give an exact meaning to "Dim→0". This is because the Dimension (not dimension!) is an $\mathbb{R}$-valued function in the framework of this theory.

14. Digression: how does the $\mathscr{C}$-spectral sequence work?

In this section we collect some results which demonstrate secondary differential forms "in action". The following two diagrams illustrate how the term $E_1(\mathcal{O})$, i.e. secondary differential forms, reflect the structure of $\mathcal{O}$. They make more precise the general estimate of the first term given by the diagram of Fig. 3. In the general case, the number of non-trivial rows on the diagram of $E_1(\mathscr{Y}_\infty)$, $\mathscr{Y} = \{F=0\}$, is equal to the highest number of non-trivial Spencer cohomology groups of the universal linearization operator l_F (see section 9).

Now we wish to discuss the meaning of some terms $E_1^{p,q}(\mathcal{O})$ of particular inter-

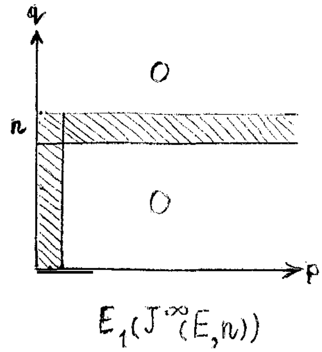

Fig. 5.

est. First of all, we have

$$E_1^{0,i}(\mathcal{O})=\bar{H}^i(\mathcal{O})$$

(see section 11) as is easily seen from definitions. In particular, it follows that elements of the term $E_1^{0,n}(\mathcal{Y}_\infty)$ are "actions" of variational problems constrained by the equation $\mathcal{Y}$ (see section 11). Moreover, if $\mathcal{L}\in E_1^{0,n}(\mathcal{Y}_\infty)$ is an action, then

$$d_1^{0,n}(\mathcal{L})=0 \tag{24}$$

is the corresponding constrained Euler–Lagrange equation. This gives a solution of the "direct" problem of the calculus of variations in the general cases of local constraints, i.e. given by means of differential equations. In particular, Eq. (24) incorporates automatically the theory of Lagrange multipliers.

The standard spectral sequence arguments applied to diagrams of the above type allow one to solve immediately the so-called *triviality problem for lagrangians* both in local and in global settings. This problem is to describe those "actions", or lagrangians, to which trivial Euler–Lagrange equations correspond. The answer, say, for free (=non-constrained) problems is that in this sense globally trivial lagrangian densities are of the form $\omega+\bar{d}\rho$, were ω is a closed differential form on $J^1(E,n)$ and $\bar{d}\rho$ is a full divergence term (see section 11).

Let now $\mathcal{Y}\subset J^k(E,n)$ be the Euler–Lagrange equations corresponding to a lagrangian $\mathcal{L}\in\bar{H}^n(J^\infty(E,n))=E_1^{0,n}(J^\infty(E,n))$. Then $E_2^{0,n}(\mathcal{Y}_\infty)$ consists of all lagrangians *alternative* to $\mathcal{L}$. To compute all alternative lagrangians one can use spectral sequence arguments as follows.

Let $\mathcal{Y}$ satisfy the conditions indicated on the diagram of Fig. 6. We also suppose that $\mathcal{Y}$ is formally integrable. Then the fact that the diagram for $E_1(\mathcal{Y}_\infty)$ consists of two non-trivial rows leads directly to the following exact sequence:

$$\begin{aligned}&0\to E_2^{1,n-1}(\mathcal{Y}_\infty)\to H^n(\mathcal{Y})\to E_2^{0,n}(\mathcal{Y}_\infty)\xrightarrow{d_2^{0,n}} E_2^{2,n-1}(\mathcal{Y}_\infty)\\&\to H^{n+1}(\mathcal{Y})\to\cdots\to E_2^{k,n}(\mathcal{Y}_\infty)\\&\xrightarrow{d_2^{k,n}} E_2^{k+2,n-1}(\mathcal{Y}_\infty)\to H^{n+k+1}(\mathcal{Y})\to\cdots,\end{aligned} \tag{25}$$

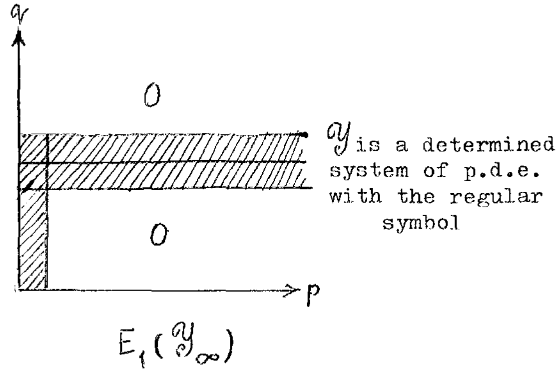

Fig. 6.

where $H^i(\mathcal{Y})$ denotes the ith de Rham cohomology group of $\mathcal{Y}$. Terms $E_2^{0,n-1}(\mathcal{Y}_\infty)$ of this sequence can be evaluated or even computed exactly by using Spencer cohomology type techniques. This gives a solution of the alternative lagrangian problem for a given $\mathcal{Y}$ as well as of a number of similar problems. The most famous of them is *the inverse problem* of the calculus of variations. This is the problem of recognizing Euler–Lagrange equations in the case when the constraints are given by $\mathcal{Y}$. Its solution is equivalent to the computation of $E_2^{1,n}(\mathcal{Y}_\infty)$, which is another term of (25). Also, the description of symplectic structures for field theories constrained by $\mathcal{Y}$ is reduced to computation of the term $E_2^{2,n}(\mathcal{Y}_\infty)$ of (25).

Going back to the first term of the $\mathcal{C}$-spectral sequence we note that the term $E_1^{0,n-1}(\mathcal{Y}_\infty)$ can be interpreted as the space of all conservation laws for solutions of $\mathcal{Y}$. This observation allows one to develop a consistent theory of conservation laws (="conserved currents"="integral of motion") independently of any symmetry considerations, which works as well in situations where the latter cannot even be applied. This theory is sketched below for formally integrable equations $\mathcal{Y}$ satisfying assumptions which guarantee the two-row diagram for $E_1(\mathcal{Y}_\infty)$ (Fig. 6).

First of all, the diagram of Fig. 6 shows that

$$H^{n-1}(\mathcal{Y})=E_2^{0,n-1}(\mathcal{Y}_\infty)\subset E_1^{0,n-1}(\mathcal{Y}_\infty)\ .$$

So, cohomology classes $\Omega\in H^{n-1}(\mathcal{Y})$ can be interpreted as conservation laws of $\mathcal{Y}$. We call them *rigid*. Also, we see that the kernel of

$$\mathrm{d}_1^{0,n-1}:E_1^{0,n-1}(\mathcal{Y}_\infty)\to E_1^{1,n-1}(\mathcal{Y}_\infty)$$

consists of these rigid conservation laws. They cannot distinguish two solutions of $\mathcal{Y}$ if one can be deformed into another. For this and some other reasons we can neglect them. On the other hand, conservation laws are uniquely characterized up to the rigid ones by their images under the differential $d_1^{0,n-1}$. This motivates us to introduce the following basic notion: the image $d_1^{0,n-1}(\Omega) \subset E_1^{1,n-1}(\mathcal{Y}_\infty)$ of a conservation law $\Omega \in E_1^{0,n-1}(\mathcal{Y}_\infty)$ is called its *generating function.*

The following isomorphism is fundamental in finding generating functions:

$$E_1^{1,n-1}(\mathcal{Y}_\infty) = \ker \bar{l}_F^*,$$

where $\mathcal{Y} = \{F=0\}$, $\bar{l}_F = l_F|_{\mathcal{Y}_\infty}$ (see section 9) and "$*$" stands for formal conjugation. So, generating functions of conservation laws can be found by solving the equation

$$\bar{l}_F^* \psi = 0 . \tag{26}$$

This is the most efficient general method of finding conservation laws for concrete equations (see Ref. [56] where it is demonstrated "in action").

Remark. In fact, not all solutions of (26) are generating functions of conservation laws. However, we can make use of the differential $d_1^{1,n-1}$ to throw away "unnecessary" solutions of (26).

Now we note that generating functions of symmetries (see section 10) and of conservation laws satisfy the mutually conjugate equations $\bar{l}_F \varphi = 0$ and $\bar{l}_F^* \psi = 0$. Also, we have $l_F = l_F^*$ for Euler–Lagrange equations. This demonstrates clearly the nature of the intimate relations between symmetries and conservation laws for Euler–Lagrange equations as given by the classical theorem of Noether. But now we see that the same relations hold for a much wider class of equations, which can be called *conformly self-adjoint*, i.e., such that

$$\bar{l}_F^* = \Lambda \circ \bar{l}_F ,$$

where Λ is an invertible operator on $\mathcal{Y}_\infty$. The equation $u_x = u_y$ is a simple example of that matter for which $\Lambda = -1$.

An interpretation of the term $E_2(\mathcal{Y}_\infty)$ is that it consists of characteristic classes of bordisms composed of solutions of $\mathcal{Y}$. The standard "differential characteristic classes" theories can be obtained in this way under a suitable choice of $\mathcal{Y}$ (see Ref. [54]). This approach leads, however, to finer characteristic classes, for instance, *special characteristic classes.* We illustrate this topic for solutions of the (vacuum) Einstein equations, or Einstein manifolds. Let $\mathcal{Y}$ be the Einstein system on a manifold M. Then it is possible to show that

$$\mathcal{Y}_\infty / \mathrm{Diffeo}(M) = \mathcal{Y}'_\infty ,$$

where Diffeo(M) is the diffeomorphism group of M, acting naturally on $\mathcal{Y}_\infty$, and $\mathcal{Y}'$ is a certain system of partial differential equations. In fact, $\mathcal{Y}'$ does not depend on M and, therefore, its solutions are just diffeomorphism classes of Einstein manifolds. The corresponding special characteristic classes are elements of $E_2(\mathcal{Y}'_\infty)$.

The $\mathcal{C}$-spectral sequence machinery was discovered by the author while trying to solve the aforementioned local and global problems of the calculus of variations and of conservations laws [46,50]. A very small number of works were published since then in this direction and we conclude by listing some of the most important ones [45,46,50,54,38,40,41,29,51].

15. Quantization or singularity propagation? Heisenberg or Schrödinger?

In the preceding pages we have reached the coasts of "terra incognita", i.e. diffieties and secondary calculus on them, whose existence was predicted by the linguisticized version of the Bohr correspondence principle as formulated in section 3. Being the exact analog of algebraic geometry for partial differential equations this branch of pure mathematics deserves to be explored systematically, maybe much more so than algebraic geometry itself and independently of the possible physical applications that stimulated the expedition. Later on we will discuss briefly some other topics related to secondary calculus. But now it would be timely to reexamine how much we have approached the solution of the quantization problem for quantum fields after having got secondary calculus at our disposal.

It should be stressed from the very beginning that the passage to the "linguisticized" version of the Bohr principle inevitably cost us the loss of its original physical context. On the other hand, the accumulated experience in secondary calculus convinces us that every *natural* construction in the area of classical Calculus has its secondary analog, which can be found by means of a more or less regular procedure. So, one can expect to deduce fundamental QFT equations by "secondarizing" a sample situation in which both the source and the target of the Bohr principle belong to the area of classical Calculus.

Evidently, quantum mechanics of particles is exactly such a sample due to the fact that the Bohr correspondence principle here starts from differential (the Schrödinger) equations and finishes also at differential (the Hamilton) equations. However, in this case the Bohr principle is to be reinterpreted exclusively in terms of Calculus to become secondarizable. This is the key point.

The desired reinterpretation is not obvious and, in particular should not be based on "$h\to 0$", formal series on H, deformations, Hilbert spaces and similar things. We accept formula (2) to be the first approximation. Then our approach to QFT can be summarized as

$$\text{FIELD QUANTIZATION} = \text{CHAR}^{-1}_{\mathcal{S}(Q)}, \tag{27}$$

where $\mathcal{S}$ stands for "secondarization". Hence, the question to be answered first is: what is the solution singularity type (or types) outlined in section 3?

The last problem belongs to the theory of solution singularities of partial differential equations, which has not been elaborated enough up to now to provide us with the immediate answer. So, we postpone the direct attack to the future and limit ourselves here to a quick trip through the theory of some special solution singularities called *geometric*. Besides all other, the reader can conceive from this model more precise ideas on the general theory as well as more detailed motivations for formula (2). But first we will permit ourselves some remarks of an historical nature.

As it is well known, two different approaches, one by Heisenberg and the other by Schrödinger, were at the origin of quantum mechanics. In modern terms, the first of them is based on a formal non-commutative deformation of the commutative algebra of classical observables while the latter proceeds from an analogy with optics. They both were proclaimed and even proved equivalent and this is just the point we would like to call in question now. Namely, it seems that a more exact formulation of this equivalence theorem would be:

The Schrödinger point of view becomes equivalent to the Heisenberg one after being reduced appropriately.

Below some brief general justifications of this assertion are given and the reader is asked not to confuse "approach" with "picture" in what follows.

First, the Heisenberg approach is "programmed" in the language of operator algebras while that by Schrödinger is in Calculus. The former is non-localizable in principle and this is its great disadvantage in what concerns applications to fundamental (non-technical!) problems of physics. In particular, the passage from one space–time domain to another cannot be expressed in terms of this language only (see, for instance, Ref. [10]). But, evidently, fundamental physical theories, both at classical and quantum levels, must be localizable in this sense by their nature. On the other hand, Calculus is the only localizable language due to the fact that localizable operators are just differential ones.

Second, in the Heisenberg approach classical mechanics appears to be a limit case of quantum mechanics or, vice versa, the latter is viewed to be a non-commutative deformation of the former. In particular, this means that they both are treated to be things of the same nature that differ from each other by a parameter. This is not so in the framework of the Schrödinger approach. In fact, as follows from the general mathematical background of the passage from wave to geometrical optics, the latter appears to be a particular aspect of the former. So, applying the analogy between quantum mechanics and optics discovered by Schrödinger one can conclude that

classical mechanics is a particular aspect of quantum mechanics.

In this connection it would be to the point to note that Planck's constant is a true constant and, therefore, "$h\to 0$" can serve as a heuristic trick but not as a groundstone of the theory.

Thus these are, shortly, the reasons in favor of the Schrödinger alternative. On the other hand, it is clearly seen that it had no chances to be realized mathematically in the building period of quantum electrodynamics and other quantum field theories. So, the Heisenberg alternative remained, due to its formality and abstraction, the only possible way for progress of these theories. This was its invaluable historical merit that seems to be going to be exhausted now. Finally, we add that this paper can be regarded also as an attempt to provide the Schrödinger approach with the mathematical tools which are necessary to extend it to QFT.

16. Geometric singularities of solutions of partial differential equations

In this section we present geometric singularities of solutions of (non-linear) partial differential equations and some general results on them that are relevant to our discussion of the quantization problem. Some examples illustrating the general theory and, in particular, the mechanism connecting wave and geometrical optics are collected in the next section.

Solution singularities which we call geometrical arise naturally in the context of the theory of multivalued solutions of (non-linear) partial differential equations. There are different ways to realize strictly the idea of multivaluedness and we choose that one which is based on the notion of R-manifold. This is as follows.

Recall that a submanifold $W\subset J^k(E,n)$, $0\leqslant k<\infty$, is called *integral* if $T_\theta W\subset C_\theta$ for every $\theta\in J^k(E,n)$ (see section 6). An integral submanifold $W\subset J^k(E,n)$ is called *locally maximal* if no open part of it belongs to another integral submanifold of greater dimension.

Definition. A locally maximal n-dimensional integral submanifold of $J^k(E,n)$ is said to be an *R-manifold.*

To motivate this definition we note that manifolds of the form $L_{(k)}$ (see section 5), which are basic for the geometric theory of partial differential equations, are completely characterized by the following two properties:

(i) $L_{(k)}$ is a locally maximal integral submanifold,

(ii) the restriction of the projection

$$\alpha_{k,k-1}: J^k(E,n)\to J^{k-1}(E,n)$$

on $L_{(k)}$ is an immersion.

So, omitting (ii) we get the multivalued analogs of submanifolds $L_{(k)}$, i.e. R-manifolds.

Remark. There exist different types of locally maximal integral submanifolds of $J^k(E, n)$ which differ from each other by their dimensions. For instance, one of these types is formed by fibers of the projection $\alpha_{k,k-1}$. These are integral submanifolds of the greatest possible dimension.

Informally R-manifolds can be treated, generally, as non-smooth n-dimensional submanifolds of E whose singularities can be resolved by lifting them onto a suitable $J^k(E, n)$.

Now we define a *multivalued solution* of a partial differential equation $\mathcal{Y} \subset J^k(E, n)$ to be an R-manifold, say W, belonging to one of its extensions $\mathcal{Y}_{(s)}$, $0 \leqslant s < \infty$: $W \subset \mathcal{Y}_{(s)} \subset J^{k+s}(E, n)$.

If $W \subset J^k(E, n)$ is an R-manifold then its *singular*, or *branch points* are defined to be the singular points of the projection $\alpha_{k,k-1}: J^k(E, n) \to J^{k-1}(E, n)$ restricted to W (see Fig. 7).

We stress here that W is a smooth (=non-singular) submanifold of $J^k(E, n)$ and the adjective "singular" refers to the projection $\alpha_{k,k-1}$.

A very rich and interesting structural theory stands behind these simple definitions. This cannot be reduced to the standard singularity (or catastrophe) theory. On the contrary, the latter is a particular degenerated case of the former.

We start with a classification of geometric singularities, which is, of course, the first structural problem to be considered. According to "the general principles" we have to classify s-jets of R-manifolds in $J^k(E, n)$ for a prescribed natural s under the group of contact transformations of this jet space. The simplest case $s = 1$ is sufficient for our purposes.

Let $W \subset J^k(E, n)$ be an R-manifold and $\theta \in \operatorname{sing} W$. The subspaces of the tangent space $T_\theta J^k(E, n)$ which are of the form $T_\theta W$ are called *singular R-planes* (at θ). So, our problem is to classify singular R-planes.

Let $P = T_\theta W$ be a singular R-plane at θ. The subspace P_0 of P which consists of vectors annihilated by $\alpha_{k,k-1}$ is called the *label of P*. It turns out that singular R-planes are equivalent iff their labels are equivalent. So, the classification problem in question is reduced to the label classification problem. We define the *type* (singular or not) of an R-plane to be the dimension of its label:

$$\operatorname{type} P = \dim P_0 , \quad 0 \leqslant \operatorname{type} P \leqslant n .$$

Evidently, type $P = 0 \Leftrightarrow P$ is non-singular.

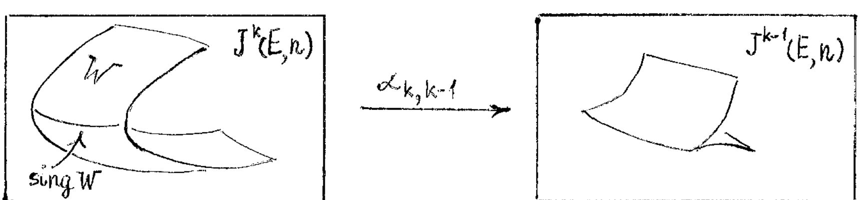

Fig. 7.

Example. Branched riemannian surfaces are identical with multivalued solutions of the classical Cauchy–Riemann equation. Let W be one of them. Then the set sing W consists of a number of isolated points, say θ_α. In this case (type P_α) $=2$ for $P_\alpha = T_{\theta_\alpha} W$.

The final result of the label classification is as follows [55].

Theorem. *Label equivalence classes of geometric singularities are in one-to-one correspondence with isomorphic classes of unitary commutative* $\mathbb{R}$*-algebra so that the dimension of a label is equal to that of the algebra corresponding to it.*

Recall that every unitary commutative finite-dimensional algebra splits into a direct sum of algebras $\mathbb{F}_{(k)}$, $k=1, 2, \ldots$, where $\mathbb{F}_{(k)}$ denotes the unitary $\mathbb{F}$-algebra generated by one element ξ such that $\xi^k=0$, $\xi^{k-1}\neq 0$ and $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Such a splitting is not unique but the multiplicity numbers showing how many times a given algebra $\mathbb{F}_{(k)}$ enters it do not depend on the splitting. So, these multiplicity numbers determine completely the isomorphism class of the algebra in question.

Below we speak of A-type geometric singularities referring to the commutative algebra A corresponding to it by the above theorem.

Examples. 1. Since the only one-dimensional $\mathbb{R}$-algebra is $\mathbb{R}$ itself, there exists only one label type of geometric singularities with a one-dimensional label. This type is realized by R-manifolds projected on the manifold of independent variables as foldings. For this reason it is denoted by FOLD. The standard theory of characteristic covectors takes a natural part in the FOLD-singularity theory.

2. There are just three isomorphic classes of two-dimensional unitary commutative algebras, namely, that of $\mathbb{C}$, $\mathbb{R}_{(2)}$ and $\mathbb{R}\oplus\mathbb{R}$ where (see above) $\mathbb{R}_{(2)}=\{1, \xi \mid \xi^2=0\}$. For equations with two independent variables $\mathbb{C}$-type geometric singularities look as ramification points of riemannian surfaces and as $(n-2)$-dimensional families of such ones for n independent variables. In four-dimensional space–time of $\mathbb{C}$-type singularity can be viewed as a vortex around a moving curve. This sets fire to the suspicion that $\mathbb{C}$-singularities could play an important role in the future turbulence theory.

The next question that arises immediately when studying concrete equations is:

What label type of geometric singularities does a given system of partial differential equations admit?

This is an essentially algebraic problem which we illustrate with the following examples to omit the general discussion.

Examples. 1. A system of partial differential equations admits FOLD-type singularities only if it admits non-zero characteristic covectors. For instance, solutions of elliptic equations do not admit FOLD-singularities.

2. Let $\mathcal{Y}$ be a second-order scalar differential equation of two independent variables. Then it admits only one of the three types of two-dimensional singularities mentioned in the preceding example. This is the $\mathbb{C}$-type for elliptic equations, the $\mathbb{R}_{(2)}$-type for parabolic equations and $\mathbb{R}\oplus\mathbb{R}$-type for hyperbolic equations.

A more delicate problem is to describe submanifolds of the form $\operatorname{sing}_\Sigma W$ for multivalued solutions W of a given differential equation and a given solution singularity type Σ. Here $\operatorname{sing}_\Sigma W \subset W$ stands for the submanifold of Σ-singular points of W. In other words, we are interested in determining the shapes of Σ-singularities admitted by a given equation.

The solution of this problem can be sketched as follows: Let a label solution singularity type Σ be fixed; then it is possible to associate with a given system of partial differential equations $\mathcal{Y}$ another system $\mathcal{Y}_\Sigma$ such that submanifolds of the form $\operatorname{sing}_\Sigma W$, W being a multivalued solution of $\mathcal{Y}$, satisfy $\mathcal{Y}_\Sigma$ and, conversely, every solution of $\mathcal{Y}_\Sigma$ is of the form $\operatorname{sing}_\Sigma W$ for (possibly formal) multivalued solutions of $\mathcal{Y}$.

If $\mathcal{Y}$ is of n independent variables, then $\mathcal{Y}_\Sigma$ is of $n-s$ independent variables where s is the dimension of the label Σ. The construction of equations $\mathcal{Y}_\Sigma$ is not simple enough to be reproduced here. Instead, in the next section we exhibit some examples from which the reader can conceive an idea of them. Informally speaking, if $\mathcal{Y}$ describes a physical substance, say a field or a continuous medium, then $\mathcal{Y}_\Sigma$ describes the behavior of a certain kind of singularities of this substance, that can be characterized by the label singularity type Σ. In the case when $\mathcal{Y}$ refers to independent space–time variables the equation $\mathcal{Y}_\Sigma$ describes the propagation of Σ-type singularities in the substance in question.

Denote by CHAR_Σ the functor that associates the equation $\mathcal{Y}_\Sigma$ with a given equation $\mathcal{Y}$. The problem:

to what extent does the behavior of the singularities of a given type of physical system determine the system itself?

is, evidently, of fundamental importance and the search for the domain of invertibility of the functor CHAR_Σ is maybe the most significant aspect of it. The following result gives an instructive example on this matter.

The FOLD-reconstruction theorem. *Every hyperbolic system of partial differential equations $\mathcal{Y}$ is determined completely by the associated system $\mathcal{Y}_{\mathrm{FOLD}}$.*

In other words, a hyperbolic system $\mathcal{Y}$ can be written down explicitly only if the system $\mathcal{Y}_{\mathrm{FOLD}}$ is known.

The above theorem can be reformulated by saying that the functor $\mathrm{CHAR}_{\mathrm{FOLD}}$ is invertible on the class of hyperbolic equations. On the other hand, this functor is not invertible on the class of elliptic equations due to the fact that $\mathcal{Y}_{\mathrm{FOLD}}$ is empty for any elliptic $\mathcal{Y}$.

The general "singularity reconstruction problem" we are discussing may have various flavors depending on the chosen, not necessarily geometric, solution singularity type. For instance, the classical problem of fields and sources can be viewed as its particular case. Another remarkable example can be found in the history of electrodynamics. Observing that the elementary laws of electricity and magnetism such as that by Coulomb or Faraday describe the behavior of some kind of singularities of electromagnetic fields, we see Maxwell's equations to give the solution of the corresponding singularity reconstruction problem.

The importance of multivalued solution theory comes in evidence also due to its relations with the Sobolev–Schwartz theory of generalized solutions of linear partial differential equations. These relations are based on the observation that one can get a generalized solution of a given linear differential equation simply by summing up branches of a multivalued solution of it. As a matter of fact, the procedure assigning the generalized solution to a given multivalued one is more delicate than a simple summation and is based on the choice of a de Rham type cohomology theory and a suitable class of test functions. Maslov-type characteristic classes then arise as obstructions to perform this procedure and their nature depends on the cohomology theory chosen (see Refs. [26,52,28]).

It is worth stressing that generalized solutions assigned to multivalued ones with no FOLD-singularities are, in fact, smooth, i.e. not properly generalized functions. This correlates nicely with the well-known fact that generalized solutions of elliptic equations are exhausted by smooth solutions, i.e. one-valued ones, while such equations admit non-trivial multivalued solutions (say, branched riemannian surfaces for the Cauchy–Riemann equation) with non-FOLD-type singularities. These and other similar facts show multivalued solutions to be a satisfactory substitution for generalized ones for non-linear differential equations where the latter cannot even be defined. Moreover, the former are a finer tool also in the framework of the linear theory.

Further details and results on the topics touched upon in this section the reader can be found in the book [17] and in the author's lecture [52]. For a systematic exposition see the forthcoming paper [55]. Many other interesting aspects of solution singularity theory are presented in the recent review by Lychagin [28].

Multivalued solutions were introduced in the author's work [43] followed by a technically simple but instructive work [18] by Krishchenko. Afterwards, a significant series of works by Lychagin appeared. Unfortunately, these names almost exhaust the list of contributors in this field. For a full bibliography see Refs. [17,28,55].

17. Wave and geometrical optics and other examples

In this section we illustrate the generalities of the previous one with some simple examples taken from Ref. [24]. We enter here neither into technical details nor into interpretations of the exhibited equation, referring the reader to Ref. [24].

17.1. Σ-characteristic equations

Let $\pi: E \to M$ be a fibering, $\mathscr{Y} \subset J^k\pi$ be a system of differential equations and Σ be a label solution singularity type. The *Σ-characteristic system* of Σ is the system of differential equations whose solutions are of the form $\pi_k(\mathrm{sing}_\Sigma W)$ where $\pi_k: J^k\pi \to M$ is the natural projection and W is a multivalued solution of $\mathscr{Y}$.

Denote the Σ-characteristic equation of $\mathscr{Y}$ by $\mathscr{Y}^0_\Sigma$ and observe that the whole system $\mathscr{Y}_\Sigma$ is obtained by adding to $\mathscr{Y}^0_\Sigma$ some other equations called *complementary*. If $\mathscr{Y}$ refers to independent space–time variables, then $\mathscr{Y}^0_\Sigma$ governs motions of Σ-singular locuses of the physical system in question while the complementary equations describe the evolution of the internal structures of Σ-singularities.

Classical characteristic equations, whose theory was initiated by Hugoniot and then developed systematically by Hadamard (see Ref. [11]), arise naturally in the study of uniqueness of the initial data problem. As we have already mentioned, the uniquenesss problem is included in the theory of FOLD singularities. So, it is not surprising that FOLD-characteristic equations coincide with classical ones. We recommend Refs. [20,21,35], in which first attempts to apply classical characteristic equations to quantum mechanics and relativity were made.

The coordinate-wise representation of FOLD-characteristic equations looks as follows. Let the basic equation $\mathscr{Y}$ be given by (10) with $F_j \in \mathscr{F}_k$, $j = 1, \ldots, l$. Introduce the *characteristic matrix* of $\mathscr{Y}$ to be

$$\mathscr{M}_F = \begin{pmatrix} \sum\limits_{|\sigma|=k} \dfrac{\partial F_1}{\partial u^1_\sigma} p^\sigma & \cdots & \sum\limits_{|\sigma|=k} \dfrac{\partial F_1}{\partial u^m_\sigma} p^\sigma \\ \vdots & & \vdots \\ \sum\limits_{|\sigma|=k} \dfrac{\partial F_l}{\partial u^1_\sigma} p^\sigma & \cdots & \sum\limits_{|\sigma|=k} \dfrac{\partial F_l}{\partial u^m_\sigma} p^\sigma \end{pmatrix},$$

where $p^\sigma = p_1^{i_1}, \ldots, p_n^{i_n}$ for $\sigma = (i_1, \ldots, i_n)$. The FOLD-characteristic equation becomes trivial, i.e. $0=0$, for $l<m$. If $l \geqslant m$ we get the FOLD-characteristic equation $\mathscr{Y}^0_{\mathrm{FOLD}}$ of Σ-singular locuses representable in the form $x_n = \varphi(x_1, \ldots, x_{n-1})$ by sub-

stituting $\partial\varphi/\partial x_i$ for p_i, $i=1, ..., n-1$, and -1 for p_n in $\mathcal{M}_F$ and then equating to zero all mth-order minors of the matrix so obtained.

Remark. Strictly speaking the above procedure is valid only for a formally integrable $\mathcal{Y}$.

17.2. Maxwell equations and geometric optics

Consider the vacuum Maxwell equations (="wave optics"):

$$\operatorname{div} E=0\,, \qquad \operatorname{rot} E=-\frac{1}{c}\frac{\partial H}{\partial t}\,,$$

$$\operatorname{div} H=0\,, \qquad \operatorname{rot} H=\frac{1}{c}\frac{\partial E}{\partial t}\,.$$

In this case $n=4$, $M=6$, $l=8$. So, the characteristic matrix is (8×6) rectangular and a direct computation shows that all its sixth-order minors are of the form

$$\lambda\left(p_1^2+p_2^2+p_3^2-\frac{1}{c^2}p_4^2\right)^2\,,$$

with $\lambda=0$ or $\pm p_i p_j$. So, putting $x_4=t$ we see that the equation $\mathcal{Y}^0_{\mathrm{FOLD}}$ coincides with the standard eikonal equation

$$\left(\frac{\partial\varphi}{\partial x_1}\right)^2+\left(\frac{\partial\varphi}{\partial x_2}\right)^2+\left(\frac{\partial\varphi}{\partial x_3}\right)^2=\frac{1}{c^2}\,. \tag{28}$$

In such a way we obtain the interpretation of this well-known fact in terms of the solution singularity theory. However, this gives us something more, namely, the complementary equations that compose together with the eikonal equation the whole system $\mathcal{Y}_{\mathrm{FOLD}}$. These look as follows:

$$\operatorname{div} h_E=\operatorname{grad}\varphi\cdot\operatorname{rot} h_H\,,$$

$$\operatorname{rot} h_E+\operatorname{div} h_H\cdot\operatorname{grad}\varphi=\operatorname{grad}\varphi\times\operatorname{rot} h_H\,.$$

Here h_E and h_H are singular values, i.e. values on the singular surface $t=\varphi(x_1, x_2, x_3)$, of the electric and magnetic fields, respectively.

17.3 On the complementary equations

It is obvious from the procedure of section 17.1 that very different equations can have the same characteristic equation. For example, the eikonal equation (28) is also the characteristic equation for the Klein–Gordon equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u - m^2 u = 0 .$$

So, it is not possible to reconstruct the original equation knowing only its characteristic equation. In view of the Reconstruction Theorem of the previous section the only information one needs for the reconstruction is contained exactly in the complementary equations. Therefore, an independent and direct physical interpretation of quantities entering into these equations would allow one to make up the information that is lacking to solve the corresponding singularity reconstruction problem. One can see now that this *singularity interpretation problem* becomes very important. For example, a solution of this problem for continuous media would provide us with a regular method

to deduce equations governing a given continuous medium proceeding from observations of how singularities of a given type (or types) propagate in it.

This would be an attractive alternative to the present phenomenological status of mechanics of continuous media.

It is clear that the quantization "à la Schrödinger" can also be treated as such a kind of interpretation problem. In this context the Hamilton–Jacobi equations of classical mechanics considered as Q-characteristic equations are to be completed by suitable complementary equations. It is natural to think that the standard formal quantization methods "à la Heisenberg" cover just the remaining gap of these hypothetical complementary equations.

17.4 Alternative singularities via the homogenization trick

The classical, i.e. FOLD-characteristic equation for the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\Delta\psi - V\psi = 0 , \tag{29}$$

and for singular locuses given in the form $t = x_4 = \varphi(x_1, x_2, x_3)$, is

$$\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2 + \left(\frac{\partial\psi}{\partial x_3}\right)^2 = 0 .$$

This demonstrates that geometric singularities are not adequate for the correspondence between quantum and classical mechanics. For the hypothetical "quantum" singularity type (see section 3) the Q-characteristic equation $\mathscr{Y}_Q^0$ should be

$$\frac{\partial\varphi}{\partial t} - \frac{1}{2m}\sum_{i=3}^{3}\left(\frac{\partial\varphi}{\partial x_i}\right)^2 - V = 0 . \tag{30}$$

It is possible, however, to interpret (30) as the classical characteristic equation for the "homogenized" Schrödinger equation,

$$\frac{\partial^2 \tilde{\psi}}{\partial t\, \partial s} + \frac{1}{2m} \sum_{i=1}^{3} \frac{\partial^2 \tilde{\psi}}{\partial x_i^2} + V \frac{\partial^2 \tilde{\psi}}{\partial s^2} = 0 \tag{31}$$

in five-space (x_1, x_2, x_3, t, s) assuming that singular locuses are given in the form $s - \varphi(x_1, x_2, x_3, t) = 0$. On the other hand, (31) reduces to (29) on the functions

$$\tilde{\psi} = \varphi(x, t) \exp((\mathrm{i}/\hbar)s) . \tag{32}$$

This motivates us to define Q-singularities as the reduction of FOLD singularities on the functions (32). This is not, however, very straightforward and we refer the reader to Ref. [24] for some results of this approach.

17.5. $\mathbb{R}_{(k)}$-characteristic equations

In this subsection some analogs of the Hamilton–Jacobi equation for extended (i.e. not point-like) singular locuses are exhibited. For simplicity we have chosen the wave equation

$$\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$$

as basic. Since $\mathbb{R}_{(1)} = \text{FOLD}$ the $\mathbb{R}_{(1)}$-characteristic equation coincides with the standard eikonal equation (28).

For $k=2$ and the singularity locuses given by

$$x_2 = \varphi(s, t) , \quad x_3 = \psi(s, t) , \qquad \text{with } s = x_1 ,$$

the $\mathbb{R}_{(2)}$-characteristic equation looks as

$$\left(\frac{\partial \psi}{\partial t} \frac{\partial \varphi}{\partial s} - \frac{\partial \varphi}{\partial t} \frac{\partial \psi}{\partial s}\right)^2 + \left(\frac{\partial \varphi}{\partial t}\right)^2 + \left(\frac{\partial \psi}{\partial t}\right)^2 - c^2 \left(\frac{\partial \varphi}{\partial s}\right)^2 - c^2 \left(\frac{\partial \psi}{\partial s}\right)^2 - c^2 = 0 .$$

Its solutions are two-dimensional surfaces tangent to the light cone.

Finally, the $\mathbb{R}_{(3)}$-characteristic equation for singularity curves of the form $x_i = x_i(t)$, $i = 1, 2, 3$, is

$$\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = c^2 .$$

For other examples, results and discussions see Ref. [24].

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